On Principal Values on P-Adic Manifolds

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In the paper [L] a project for proving the existence of transfer factors for forms of $SL(3)$, especially for the unitary groups studied by Rogawski, was begun, and it was promised that it would be completed by the present authors. Their paper is still in the course of being written, but the present essay can serve as an introduction to it. It deals with $SL(2)$ which has, of course, already been dealt with systematically [L-L], the existence of the transfer factors being easily verified. Thus it offiers no new results, but develops, in a simple context, some useful methods for computing the principal value integrals introduced in [L].

We describe explicitly the Igusa fibering, form and integrand associated to orbital integrals on forms of $SL(2)$, taking the occasion to clarify the relation of this fibering to the Springer-Grothendieck resolution (cf. §3). The Igusa data established, there are two problems: (i) to show that certain principal values are zero, (ii) to compare principal values on two twisted forms of the same variety. To deal with the first we have, in §1, computed directly some very simple principal values on $\mathbf{P}^1$, and shown that principal values behave like ordinary integrals under standard geometric operations such as fibering and blowing-up. The second problem is dealt with in a similar way, by using Igusa's methods to establish, in a simple case, a kind of comparison principle (Lemma 4.B).

The endoscopic groups for a form of $SL(2)$ are either tori or $SL(2)$. For tori the solution of the first problem (Lemma 4.A) leads immediately to the existence of transfer factors, and the hypotheses of [L_1, pp. 102, 149] are trivially satisfied. If G is anisotropic over F and the endoscopic group is $SL(2)$ the solution of the second problem (Lemma 4.B with $\kappa \equiv 1$) and the characterization of stable orbital integrals (cf. [V]) yields the existence of transfer factors as well as the local hypothesis of [L_1, p. 102]. The analogous results at archimedean places are known in general (cf. [L_1, Lemma 6.17]). The global hypothesis [L_1, p. 149] follows from [L_1, Lemma 7.22].

The principal values which arise for forms of $SL(2)$ are computed without difficulty, but we expressly avoid such calculations. The aim of the project begun in [L], and continued here, is to develop methods for proving the existence of transfer factors which appeal only to geometric techniques of some generality and thus have some prospect of applying to all groups. One encouraging sign is the smoothness with which they mesh with the notion of κ-orbital integral. They can be easily applied to the study of the germ at regular unipotent elements. A further test, perhaps not easy to carry out, would be the semi-regular elements, already studied for $GL(n)$ by Repka [R].

Throughout this paper F will be a nonarchimedean local field of characteristic zero, with residue field of q elements; $|\cdot|_F = |\cdot|$ will denote the valuation on F and ϖ a prime element; $\bar{F}$ will be an algebraic closure of F.

§1. **Remarks.**

The following lemmas concern the simplest of the principal value integrals which arise in §1 of [L].

Let $N = N(m_1, \ldots, m_n)$ be the box

$$|u_j| \leq q^{-m_j} \qquad (1 \leq j \leq n) \tag{1.1}$$

in F^n. Consider the (multi-valued) differential form

$$\nu_{(c_1,\ldots,c_n)} \prod_{j=1}^{n} u_j^{c_j} \frac{du_1}{u_1} \wedge \ldots \wedge \frac{du_n}{u_n}, \tag{1.2}$$

where $c_1, \ldots, c_n$ are rational numbers. Let $\theta_1, \ldots, \theta_n$ be quasicharacters on $F^\times$. Writing

$$\theta_j = \theta_j |\cdot|^{t_j} \tag{1.3}$$

with θ_j unitary and g_j a real number, we assume

$$t_j + c_j \neq 0 \text{ if } \theta_j \equiv 1 \qquad (1 \leq j \leq n). \tag{1.4}$$

Set

$$h_{(\theta_1,\ldots,\theta_n)}(u_1, \ldots, u_n) = \prod_{j=1}^{n} \theta_j(u_j). \tag{1.5}$$

Then (1.4) allows us to define the principal value integral

$$\oint_N h_{(\theta_1,\ldots,\theta_n)} |\nu_{(c_1,\ldots,c_n)}| \tag{1.6}$$

following [L, Lemma 1.3]. Thus consider for $\mathrm{Re}(\mathrm{s_j}) \gg 0$ $(1 \le \mathrm{j} \le \mathrm{n})$

$$\begin{aligned}
\int_N \prod_{j=1}^n |u_j|^{s_j} h_{(\theta_1,\dots,\theta_n)} |\nu_{(c_1,\dots,c_n)}| &= \prod_{j=1}^n \int_{|u_j| \le q^{-m_j}} \theta_j(u_j)|u_j|^{s_j+t_j+c_j} \frac{du_j}{|u_j|} \\
&= \prod_{j=1}^n \sum_{n-m_j}^{\infty} \left(\int_{|u_j|=q^{-n}} \theta_j(u_j) \frac{du_j}{|u_j|} \right) q^{-(s_j+t_j+c_j)n} \qquad (1.7) \\
&= \epsilon(1-\frac{1}{q})^n \prod_{j=1}^n \frac{(\theta_j(\varpi)q^{-(s_j+t_j+c_j)})^{m_j}}{1-\theta_j(\varpi)q^{-(s_j+t_j+c_j)}},
\end{aligned}$$

where

$$\epsilon = \begin{cases} 1 \text{ if each } \theta_j \text{ is unramified} \\ 0 \text{ otherwise.} \end{cases}$$

The analytic continuation of this function is, thanks to (1.4), analytic at $s_1 = \ldots = s_n = 0$; (1.6) is the value at $s_1 = \ldots = s_n = 0$. Thus:

Lemma 1.A.

$$\oint_{N(m_1,\dots,m_n)} h_{(\theta_1,\dots,\theta_n)} |\nu_{(c_1,\dots,c_n)}| = \epsilon(1-\frac{1}{q})^n \prod_{j=1}^n \frac{(\theta_j(\varpi)q^{-(t_j+c_j)})^{m_j}}{1-\theta_j(\varpi)q^{-(t_j+c_j)}}.$$

To define now $\oint_X h|\nu|$ we assume:

(1.8) X is an F-manifold, h is a $\mathbf{C}$-valued function supported on a compact open subset of X, ν is a differential form on X; and

(1.9) the support of h is the disjoint union of neighborhoods $\mathbf{U}$ with the following properties:

(1.10) there are local coordinates $u_1, \ldots, u_n$ on X such that $\mathbf{U}$ is given by (1.1) for some $m_1, \ldots, m_n$,

(1.11) on $\mathbf{U}, \nu = \alpha\nu_{(c_1,\dots,c_n)}$ with $|\alpha|$ constant, and $h = \gamma h_{(\theta_1,\dots,\theta_n)}$ with γ constant, where $c_1, \ldots, c_n$ and $\theta_1, \ldots, \theta_n$ satisfy (1.4).

Then

$$\oint_X h|\nu| \overset{\text{def.}}{=} \sum_{\mathbf{U}} \gamma|\alpha| \oint_{N(m_1,\dots,m_n)} h_{(\theta_1,\dots,\theta_n)} |\nu_{(c_1,\dots,c_n)}|.$$

Our definition is that for the case $r = s = 1$ in the proof of Proposition 1.2 in [L]. The integral is independent of the choice for $\{\mathbf{U}, u_1, \ldots, u_n\}$ ([L, Proposition 1.2]). Note especially that the conditions (1.9) – (1.11) are local, i.e., they are satisfied if we can find around each point in the support of h a neighborhood $\mathbf{U}$ satisfying (1.10) and (1.11). We will allow α to take values in a finite Galois extension L of F; in that case, $|\alpha| = |\mathrm{Nm}_F^L \alpha|^{1/[L:F]}$.

The following remark will simplify a later argument. Given X, h and ν as in (1.8), a patch $\mathbf{U}$ as in (1.9), and integers $M_j (1 \leq j \leq r)$ such that $M_j \geq m_j$, let $\bar{\mathbf{U}} = \bar{\mathbf{U}}(M_1, \ldots, M_r)$ be the subset

$$|u_j| = q^{-M_j}(1 \leq j \leq r), |u_j| \leq q^{-m_j} \quad (r+1 \leq j \leq n)$$

of $\mathbf{U}$. Then:

Lemma 1.B.

$\oint\limits_{\bar{\mathbf{U}}} h|\nu|$ *exists and equals the value at* $s_1 = \ldots = s_n = 0$ *of*

$$\gamma|\alpha| \int\limits_{\bar{\mathbf{U}}} \prod_{j=1}^{n} |u_j|^{s_j} h_{(\theta_1,\ldots,\theta_n)} |\nu_{(c_1,\ldots,c_n)}| \quad .$$

Moreover, if the support of h is the disjoint union of a collection $\mathbf{S}$ of such neighborhoods then

$$\oint\limits_X h|\nu| = \sum_{\bar{\mathbf{U}} \in \mathbf{S}} \oint\limits_{\bar{\mathbf{U}}} h|\nu| \quad .$$

Proof: The first assertion follows from the definitions, and the second from the independence of $\oint\limits_X h|\nu|$ from the choice of decomposition for the support of h.

We consider an example. Let $U_0, \ldots, U_n$ be homogeneous coordinates on $\mathbf{P}^n$. Suppose that $\theta_0, \ldots, \theta_n$ are quasicharacters on $F^\times$ such that $\prod\limits_{j=1}^{n} \theta_j \equiv 1$, and that $c_0, \ldots, c_n$ are rational numbers such that $\sum\limits_{j=0}^{n} c_j = 0$. Assume

(1.12) $t_j + c_j \neq 0$ if $\theta_j \equiv 1 (0 \leq j \leq n)$, where $\theta_j = \theta_j |\cdot|^{t_j}$.

Let ν be the form on $\mathbf{P}^n$ given on $U_k \neq 0$ by

$$(-1)^k \prod_{j=0}^{n} U_j^{c_j} \frac{dU_0}{U_0} \wedge \ldots \wedge \frac{\widehat{dU_k}}{U_k} \wedge \ldots \wedge \frac{dU_n}{U_n}, \tag{1.13}$$

where $\widehat{\ }$ indicates deletion. Let h be the function

$$h(U_0, \ldots, U_n) = \prod_{j=0}^{n} \theta_j(U_j). \tag{1.14}$$

Then (1.12) ensures that $\oint\limits_{\mathbf{P}^n(F)} h|\nu|$ is well-defined.

Lemma 1.C.

$$\oint_{\mathbf{P}^n(F)} h|\nu| = 0.$$

Proof for $n = 1$: Set $\theta = \theta_0, t = t_0, c = c_0, u = U_0$, on $U_1 = 1$ and $u = U_1$ on $U_0 = 1$. Then

$$\begin{aligned}\oint_{\mathbf{P}^1(F)} h|\nu| &= \oint_{|u|\le 1} \theta(u)|u|^{t+c}\frac{du}{|u|} + \oint_{|u|\le q^{-1}} \theta^{-1}(u)|u|^{-(t+c)}\frac{du}{|u|} \\ &= \epsilon(1-\frac{1}{q})\left(\frac{1}{1-\theta(\varpi)q^{-(t+c)}} + \frac{\theta(\varpi)^{-1}q^{t+c}}{1-\theta(\varpi)^{-1}q^{t+c}}\right) \qquad (1.15)\\ &= 0.\end{aligned}$$

The proof for $n > 1$ will be by reduction to the case $n = 1$; it follows Lemma 1.F.

Consider X, h and ν as in (1.8) – (1.11) and a neighborhood $\mathbf{U}$ as in (1.9). To compute $\oint_{\mathbf{U}} h|\nu|$ we may change coordinates and assume that $m_1 = \ldots = m_n = 1$. This will be done for the next lemma.

Suppose that we blow up X at $u_1 = \ldots = u_n = 0$ to obtain the F-manifold $\bar{X}$ and projection $\pi : \bar{X} \to X$. Let $\bar{\mathbf{U}} = \pi^{-1}(\mathbf{U}), \bar{h} = h \circ \pi$ and $\bar{\nu} = \pi^*(\nu)$.

Lemma 1.D.

Assume that $\sum_{j=1}^{n}(t_j + c_j) \neq 0$ *if* $\prod_{j=1}^{n}\theta_j \equiv 1$. *Then* $\oint_{\bar{\mathbf{U}}} \bar{h}|\bar{\nu}|$ *exists and equals* $\int_{\mathbf{U}} h|\nu|$.

Proof: Near $u_1 = \ldots = u_n = 0, \bar{X}$ is given by $u_iU_j = U_iu_j (i, j = j, \ldots, n)$, where $U_1, \ldots, U_n$ are homogeneous coordinates on $\mathbf{P}^{n-1}(F)$.

On $U_i = 1$ we have the coordinates $U_1, \ldots, U_{i-1}, z = u_i, U_{i+1}, \ldots, U_n$. Then $u_j = zU_j (j \neq i)$. Thus

$$\bar{\nu} = \alpha z^{(c_1+\ldots+c_n)} \prod_{j\neq i} U_j^{c_j} \frac{dU_1}{U_1} \wedge \ldots \wedge \frac{dz}{z} \wedge \ldots \frac{dU_n}{U_n}$$

and

$$\begin{aligned}\bar{h} &= \gamma \prod_{j=1}^{n}\theta_j(z)\prod_{j\neq i}\theta_j(U_j) \\ &= \gamma|z|^{(t_1+\ldots+t_n)}\prod_{j\neq i}|U_j|^{t_j}\prod_{j=1}^{n}\theta_j(z)\prod_{j\neq i}\theta_j(U_j)\end{aligned}$$

on this patch.

Let $x \in \mathbf{U}$ have coordinates $u_1, \ldots, u_n$. Set $S(x) = \{i : |u_i| = \max_{1\le j\le n}|u_j|\}$. For $S \subseteq \{1, 2, \ldots, n\}$, let $\mathbf{U}_S = \{x \in \mathbf{U} : S(x) = S\}$. Then $\mathbf{U}$ is the disjoint union of the $\mathbf{U}_S$. Let $\bar{\mathbf{U}}_S = \pi^{-1}(\mathbf{U}_S)$. Fix $i \in S$.

Then $\bar{\mathbf{U}}_S$ is contained in $U_i = 1$; it consists of the points in $U_i = 1$ with $|z| \leq 1, |U_j| = 1 \ (j \in S, j \neq i)$ and $|U_k| < 1 (k \notin S)$. The assumption in the statement of the lemma ensures that $\oint_{\bar{\mathbf{U}}_S} \bar{h}|\bar{\nu}|$ and $\oint_{\bar{\mathbf{U}}} \bar{h}|\bar{\nu}|$ are well-defined.

On the other hand, $\oint_{\mathbf{U}} h|\nu|$ is the value at $s_1 = \ldots = s_n = 0$ of

$$\int_{\mathbf{U}} \prod_{j=1}^{n} |u_j|^{s_j} h|\nu| = \sum_{S} \int_{\mathbf{U}_S} \prod_{j=1}^{n} |u_j|^{s_j} h|\nu|$$
$$= \sum_{S} \int_{\bar{\mathbf{U}}_S} |z|^{s_1+\ldots+s_n} \prod_{j\neq i} |U_j|^{s_j} \bar{h}|\bar{\nu}|,$$

where for each $S \subseteq \{1, 2, \ldots, n\}$ we have fixed $i \in S$. Then by Lemma 1.B,

$$\oint_{\mathbf{U}} h|\nu| = \sum_{S} \oint_{\bar{\mathbf{U}}_S} \bar{h}|\bar{\nu}| = \oint_{\bar{\mathbf{U}}} \bar{h}|\bar{\nu}|,$$

and we are done.

Corollary 1.E. *Under the assumption of Lemma 1.D,* $\oint_{\bar{X}} \bar{h}|\bar{\nu}|$ *is well-defined and equals* $\oint_{X} h|\nu|$.

Lemma 1.F. *Suppose that* $\phi : X \to X'$ *is a smooth (submersive) map of* F*-manifolds. Suppose that* X, h *and* ν *satisfy (1.9) – (1.11) with the further constraint on the coordinates* $u_1, \ldots, u_n$*:*

(1.16) there are coordinates $v_1, \ldots, v_r$ *on* $\phi(\mathbf{U})$ *such that* $u_j = v_j \circ \phi, j = 1, \ldots, r$.

Let ν' *be a differential form on* X' *given on* $\phi(\mathbf{U})$ *by*

$$\alpha' \prod_{j=1}^{r} v_f^{d_j} \frac{dv_1}{v_1} \wedge \ldots \wedge \frac{dv_r}{v_r}, \tag{1.17}$$

where $|\alpha'|$ *is constant and* d_j *is rational,* $1 \leq j \leq r$*. Then for* $x' \in X'(F)$ *the principal value integral*

$$H(x') = \oint h \frac{|\nu|}{|\phi^*(\nu')|},$$

taken over the fiber above x' *in* X*, is well-defined outside a locally finite family of divisors. Moreover,* $\oint_{X'} H|\nu'|$ *is well-defined and*

$$\oint_{X} h|\nu| = \oint_{X'} H|\nu'|.$$

Proof: We may assume that the support of h is contained in a neighborhood $\mathbf{U}$ as in the statement of the lemma. Let $x' \in \phi(\mathbf{U})$ have coordinates $v_1, \ldots, v_r$. The fiber integral

$$H(x') = \gamma|\alpha||\alpha'|^{-1} \prod_{j=1}^{r} \theta_j(u_j)|u_j|^{c_j - d_j} \oint \prod_{j=r+1}^{n} \theta_j(u_j)|u_j|^{c_j} \frac{du_{1+1}}{|u_{r+1}|} \cdots \frac{du_n}{|u_n|}$$

is well-defined provided none of $v_1, \ldots, v_r$ vanish at x'. Then

$$\oint H|\nu'| = \gamma|\alpha| \oint \prod_{j=r+1}^{n} \theta_j(u_j)|u_j|^{c_j} \frac{du_{r+1}}{|u_{r+1}|} \cdots \frac{du_n}{|u_n|} \oint \prod_{j=1}^{r} \theta_j(u_j)|u_j|^{c_j} \frac{du_1}{|u_1|} \cdots \frac{du_r}{|u_r|}.$$

is well-defined and coincides with $\oint h|\nu|$. Thus the lemma is proved.

Proof of Lemma 1.C:

Let p be the point in $\mathbf{P}^n(F)$ where $U_0 = U_1 = \cdots = U_{n-1} = 0$. Suppose that we blow up $\mathbf{P}^n$ at p to obtain the smooth variety Q over F. The local conditions of Lemma 1.D are met since

$$\sum_{j=0}^{n-1} (t_j + c_j) = -(t_n + c_n) \text{ and } \prod_{j=0}^{n-1} \theta_j = \theta_n^{-1}$$

(cf. (1.12)). Corollary 1.E then implies that $\oint\limits_{\mathbf{P}^n(F)} h|\nu| = \oint\limits_{Q(F)} \bar{h}|\bar{\nu}|$. We define a smooth map $\phi^0 : \mathbf{P}^n - \{p\} \longrightarrow \mathbf{P}^{n-1}$ by mapping the point with homogeneous coordinates $U_0, \ldots, U_n$ in $\mathbf{P}^n$ to the point with homogeneous coordinates $U_0, \ldots, U_{n-1}$ in $\mathbf{P}^{n-1}$. There is a smooth extension $\phi : Q \longrightarrow \mathbf{P}^{n-1}$ of ϕ^0, with fiber $\mathbf{P}^1$. An easy calculation verifies that the conditions of Lemma 1.F are met and that the integral $H(x')$ over the fiber above $x' \in \mathbf{P}^{n-1}(F)$ takes the form (1.15). But then $H \equiv 0$. We conclude that $\oint\limits_{Q(F)} \bar{h}|\bar{\nu}| = 0$, and the lemma is proved.

Finally, there are two remarks which will be useful for the proof of Lemma 4.B. We state them only in the generality needed for that lemma.

Remark 1.G.

Let $L \subset \bar{F}$ be a quadratic extension of F. Denote the natural action of the nontrivial element σ of $\mathrm{Gal(F/F)}$ by a bar. We define a twisted form S of $\mathbf{P}^1$ by requiring that σ act on the homogeneous coordinates U_0, U_1 by $U_0 \longrightarrow U_1, U_1 \longrightarrow U_0$. Then $S(F)$ is contained in the affine patch $U_1 \neq 0$ and is given by $u\bar{u} = 1$ if we require $U_1 = 1$ and set $U_0 = u$. The form ν on $\mathbf{P}^1(L) = S(L)$ given by (1.13) with $c_1 = c_2 = 0$ is preserved by the Galois action of S; $|\nu| = \frac{du}{|u|}$ is a Haar measure on $S(F)$. Thus, for any character θ on $\{u \in L^\times : u\bar{u} = 1\}$, $\int\limits_{S(F)} \theta(u)\frac{du}{|u|}$ exists as an ordinary integral and is zero unless θ is trivial.

Remark 1.H.

Again L will be a quadratic extension of F. We regard $\mathbf{P}^1(L)$ as the F-rational points on a twisted form R of $\mathbf{P}^1 \times \mathbf{P}^1$ as follows: $R(L) = \mathbf{P}^1(L) \times \mathbf{P}^1(L)$ and σ acts by $(p, q) \longrightarrow (\bar{q}, \bar{p})$, so that $R(F) = \{(p, \bar{p}) : p \in \mathbf{P}^1(L)\}$. Define a form on $R(L)$ by $\nu = \frac{du \wedge dv}{uv}$, where u (respectively, v) denotes the coordinate U_0 on $U_1 = 1$ in the first (respectively, second) copy of $\mathbf{P}^1(L)$. At a point of $R(F)$ on $(U_1 = 1) \times (U_1 = 1)$ we have $v = \bar{u}$. Let h be given at such a point by $\theta(u\bar{u})|u\bar{u}|^t$, where θ is a character on $F^\times$ and t is a real number such that $t \neq 0$ if $\theta^2 \equiv 1$. Observe that, in general, h and ν do not satisfy the conditions of (1.9) - (1.11). We may, however, blow up R at $u = v = 0$ to obtain a variety $\bar{R}$ over F and projection $\pi : \bar{R} \longrightarrow R$. Set $\bar{h} = h \circ \pi$ and $\bar{\nu} = \pi^*(\nu)$. Let $\bar{N}$ be the inverse image in $\bar{R}(F)$ of the neighborhood $|u|_L \leq 1$ of $u = v = 0$ in $R(F)$. Then a calculation with coordinates shows that

$$(1.18) \qquad \oint\limits_{\bar{N}} \bar{h}|\bar{\nu}| \text{ is well-defined}$$

(here $t \neq 0$ if $\theta^2 \equiv 1$ is needed) and

$$(1.19) \qquad \oint\limits_{N} \bar{h}|\bar{\nu}| = \oint\limits_{|u|_L \leq 1} \theta \circ \mathrm{Nm}(u)|u|_L^t \frac{d_L u}{|u|_L},$$

where the subscript L indicates that we are computing on the L-manifold $|u|_L \leq 1$ in $\mathbf{P}^1(L)$. Observe that if θ is trivial on $\mathrm{Nm}_F^L L^\times$ and $t = -1$ then

$$\oint\limits_{\bar{R}(F) - \bar{N}} \bar{h}|\bar{\nu}| = \oint\limits_{R(F) - N} h|\nu|$$

is well-defined and equals

$$\oint\limits_{|u|_L < 1} d_L u = \oint\limits_{|u|_L > 1} \frac{d_L u}{|u|_L^2}.$$

Thus, in this case, we have

$$(1.20) \qquad \oint\limits_{\bar{R}(F)} \bar{h}|\bar{\nu}| = \oint\limits_{\mathbf{P}^1(L)} \frac{d_L u}{|u|_L^2} = 0 \quad \text{(Lemma 1.C)} .$$

§2. **Igusa Theory.**

Recall the setting of [L, §1]: Y is a smooth variety over $F; \phi : Y \longrightarrow C$ is an Igusa fibering of Y over a smooth curve C over F; ω is an Igusa form on Y; and f an Igusa integrand (the definitions will be reviewed presently). There is a distinguished point c_0 on $C(F)$ and ϕ is smooth except on the special fiber $\phi^{-1}(c_0)$. Choose an F-coordinate λ around c_0 on C; assume $\lambda(c_0) = 0$. Then Igusa's theory establishes the existence of an asymptotic expansion

$$\sum_{(\theta,\beta,r)} \theta(\lambda)|\lambda|^{\beta-1}(-\log_q|\lambda|)^{r-1}F_r(\theta,\beta,f)$$

near $\lambda = 0$ for the integral

$$F(\lambda) = \int f \frac{|\omega|}{|\phi^*(d\lambda)|}$$

over the fiber in $Y(F)$ above the point in C with coordinate λ. Here θ denotes a character on $F^\times, \beta$ a real number and r a positive integer. The coefficients $F_r(\theta,\beta,f)$ are the principal value integrals of [L, Proposition 1.2]. Under an assumption we will make (2.9), only $r = 1$ occurs and $F_1(\theta,\beta,f)$ is an integral of the type considered in the last section. In this paragraph we will relax the constraints on the form ω and integrand f. The fiber integal $F(\lambda)$ may then exist only as a principal value integral, but it will still have an asymptotic expansion. The coefficients are again given by [L, Proposition 1.2], i.e., by (related) principal value integrals.

For the rest of this section we require the following of Y, C, ϕ, ω and f:

(2.1) Y is a smooth variety over F, C is a smooth curve over F with distinguished point $c_0 \in C(F), \phi : Y \longrightarrow C$ is an F-morphism smooth except over c_0, ω is a differential form of maximal degree on Y, f is a $\mathbf{C}$-valued function supported on a compact open subset of $Y(F)$; **and**

(2.2) if $y_0 \in Y(F)$ lies over the coordinate patch for λ, a fixed local F-coordinate around c_0 on C, then there exist local F-coordinates $\mu_1, \ldots, \mu_n$ around y_0 on Y such that:

(2.3) if $y_0 \in \phi^{-1}(c_0)$ then ϕ is given near y_0 by $\lambda = \alpha\mu_1^{a_1} \ldots \mu_n^{a_n}$, where α is regular and invertible at y_0 and $a_1, \ldots, a_n$ are nonnegative integers; if $y_0 \notin \phi^{-1}(c_0)$ and λ_0 is the coordinate of $\phi(y_0)$ then $\mu_1 = \lambda - \lambda_0$;

(2.4) ω is given near y_0 by

$$W \prod_{j=1}^{n} \mu_j^{b_j} \frac{d\mu_1}{\mu_1} \wedge \ldots \wedge \frac{d\mu_n}{\mu_n},$$

where W is regular and invertible at y_0 and $b_1, \ldots, b_n$ are rational numbers; if $y_0 \notin \phi^{-1}(c_0)$ then $b_1 = 1$;

(2.5) f is given on points of $Y(F)$ near y_0 by $\gamma K_1(\mu_1)\dots K_n(\mu_n)$, where γ is locally constant around y_0 and $K_1,\dots,K_n$ are quasicharacters on $F^\times$ such that:

(2.6) if $\mu_j = 0$ is the branch of a divisor E in $\phi^{-1}(c_0)$ through y_0 then K_j depends only on E; if $y_0 \notin \phi^{-1}(c_0)$ then $K_j \equiv 1$; and

(2.7) if $K_j = \kappa_j|\cdot|^{t_j}$ with κ_j unitary and g_j real then either $t_j + b_j \neq 0$ or $k_j \not\equiv 1, 1 \leq j \leq n$. For Lemma 2.A, (2.7) need only be satisfied for $y_0 \notin \phi^{-1}(c_0)$.

Remark. These are the conditions of [L, §1] for $\phi : Y \longrightarrow C$ to be an Igusa fibering; ω is an Igusa form if $b_1,\dots,b_n$ are positive integers, i.e., ω has no singularities, and the zeros ω lie on the special fiber, i.e., $b_j = 0$ unless $\mu_j = 0$ is the branch of a divisor in $\phi^{-1}(c_0)$: f is an Igusa integrand if $K_1,\dots K_n$ are unitary and $K_j \equiv 1$ unless $\mu_j = 0$ is the branch of a divisor in $\phi^{-1}(c_0)$.

Let $\mathfrak{E}$ be the set of all divisors in $\phi^{-1}(c_0)$ meeting the support of f. If $\mu_j = 0$ is the branch of $E \in \mathfrak{E}$ through y_0 then $a_j = a(E)$, the multiplicity of E in $\phi^{-1}(c_0)$. We then also set $b_j = b(E), K_j = K(E), \kappa_j = \kappa(E)$ and $t_j = t(E)$, as our assumptions allow.

Let

$$F(\lambda) = \oint f \frac{|\omega|}{|\phi^*(d\lambda)|},$$

the integral being taken over the fiber in $Y(F)$ above the point on $C(F)$ with coordinate $\lambda \neq 0$. Then $F(\lambda)$ is a well-defined principal value integral of the type studied in the last section. To check this we may assume that f is supported on a neighborhood $|\mu_j| \leq \epsilon_j, 1 \leq j \leq n$, in a coordinate patch (2.2) around $y_0 \notin \phi^{-1}(c_0)$. We may also assume $|W|$ and γ constant. Then

$$F(\lambda) = \gamma|W| \oint_{\substack{|u_j|<\epsilon_j \\ (j>1)}} \prod_{j>1} K_j(\mu_j)|\mu_j|^{b_j} \frac{d\mu_2}{|\mu_2|} \cdots \frac{d\mu_n}{|\mu_n|}. \tag{2.8}$$

and we are done.

The data for the asymptotic expansion of $F(\lambda)$ will be a slight modification of that of [L, Proposition 1.1]. Consider pairs (θ, β), where θ is a character on $F^\times$ and β is a real number. Let $\mathfrak{E}(\theta,\beta)$ be the set of those $E \in \mathfrak{E}$, i.e., of those divisors E in $\phi^{-1}(c_0)$ meeting the support of f, such that $k(E) = \theta^{a(E)}$ and

$$\beta(E) \overset{\text{def.}}{=} \frac{b(E) + t(E)}{a(E)} = \beta \quad .$$

Let $e(\theta,\beta)$ be the maximum number of branches of divisors in $\mathfrak{E}(\theta,\beta)$ meeting at a point. For the purposes of this paper it will be sufficient to consider the case:

$$e(\theta,\beta) \le 1 \quad . \tag{2.9}$$

Then:

Lemma 2.A.

For $|\lambda|$ sufficiently small,

$$F(\lambda) = \sum_{(\theta,\beta)} \theta(\lambda)|\lambda|^{\beta-1} F_1(\theta,\beta,f)$$

where $F_1(\theta,\beta,f)$ is the constant of [L, Proposition 1.2].

If $e(\theta,\beta)=0$ then $F_1(\theta,\beta,f)=0$. Otherwise, let E be a divisor in $\mathfrak{E}(\theta,\beta)$. Suppose that $y_0 \in E(F)$. Choose coordinates $\mu_1,\dots,\mu_n$ as in (2.2) and assume that $\mu_1=0$ is a branch of E through y_0. Following [L, Proposition 1.2] we define h and ν near y_0 by

$$h = h(\mu_2,\dots,\mu_n) = \frac{\gamma(0,\mu_2,\dots,\mu_n)}{\theta^\beta(\alpha(0,\mu_2,\dots,\mu_n))} \prod_{j=2}^{n} K_j(\mu_j)\theta(\mu_j^{-a_j})|\mu_j|^{-\beta a_j} \ ,$$

where $\theta^\beta = \theta|\cdot|^\beta$, and

$$\nu = W(0,\mu_2,\dots,\mu_n)\prod_{j=2}^{n}\mu_j^{b_j}\frac{d\mu_2}{\mu_2}\wedge\dots\wedge\frac{d\mu_n}{\mu_n}.$$

Then

$$F_1(\theta,\beta,f) = \sum_E \oint_{E(F)} h|\nu|, \tag{2.10}$$

These integrals to be calculated by the methods of §1.

Proof of Lemma 2.A:

We may assume that f is supported on a coordinate patch (2.2) around $y_0 \in \phi^{-1}(c_0)$. Then f and ω, and hence $F(\lambda)$, come with the parameters $t=(t_1,\dots,t_n)$ and $b=(b_1,\dots,b_n)$. We write $F(\lambda)=F(\lambda,t,b)$.

If $t_j+b_j \ge 1$ $(1\le j\le n)$ then arguments of [L, Propositions 1.1 and 1.2] carry through without modification, for $F(\lambda,t,b)$ is an ordinary integral. Thus the lemma is proved in this case.

We now relax this condition on t and b. Let $t' = (t'_1, \dots, t'_n) \in \mathbf{R}^n$. It is convenient to assume that

$$\frac{t'}{a_i} = \frac{t'_j}{a_j} \text{ if } a_i, a_j \neq 0.$$

Suppose that $E \in \mathfrak{E}$ has data (θ, β) with respect to (t, b), i.e., with respect to f and ω. If $\mu_j = 0$ is a branch of E through y_0 then E has data $\theta' = \theta$ and

$$\beta' = \frac{b_j + t_j + t'_j}{a_j} = \beta + \frac{t'_j}{a_j}$$

with respect to $(t + t', b)$. Thus (2.9) is satisfied by $(t + t', b)$. If $t'_j \gg 0, 1 \leq j \leq n$, then $t_j + t'_j + b_j \geq 1, 1 \leq j \leq n$, and there exists $\epsilon > 0$ independent of t' such that

$$F(\lambda, t + t', b) = \sum_{(\theta', \beta')} \theta'(\lambda)|\lambda|^{\beta' - 1} F_1(\theta', \beta', f) \tag{2.11}$$

for $|\lambda| < \epsilon$. One verifies easily that $F(\lambda, t, b)$ is the value at $t' = 0$ of $F(\lambda, t + t', b)$. At $t' = 0$ the right side of (2.11) has the value

$$\sum_{(\theta, \beta)} \theta(\lambda)|\lambda|^{\beta - 1} F_1^0(\theta, \beta', f)$$

where $F_1^0(\theta, \beta', f)$ is the value of $F_1(\theta, \beta', f)$ at $t' = 0$. This is readily seen to be $F_1(\theta, \beta, f)$, and the lemma is proved.

The following remarks will not be needed in this paper.

Let $\epsilon > 0, I(\epsilon) = \{c \in C(F) : |\lambda| = |\lambda(c)| \leq \epsilon\}$ and $Y(\epsilon)$ be the inverse image of $I(\epsilon)$ in $Y(F)$. The asymptotic expansion for $F(\lambda)$ allows us to define the principal value integral $\oint_{I(\epsilon)} F(\lambda) d\lambda$ as the value of $\int_{I(\epsilon)} F(\lambda)|\lambda|^s d\lambda$ at $s = 0$ provided $F_1(1, 0, f) = 0$, i.e., provided there is no contribution from the pair $\theta \equiv 1, \beta = 0$ to the expansion. On the other hand our initial assumptions ensure that $\oint_{Y(\epsilon)} f|\omega|$ is well-defined (in the sense of §1).

Lemma 2.B. *Assume $F_1(1, 0, f) = 0$. Then*

$$\oint_{Y(\epsilon)} f|\omega| = \oint_{I(\epsilon)} F(\lambda) d\lambda \quad .$$

Proof: We may assume that f is supported on a neighborhood in a coordinate patch (2.2) around $y_0 \in \phi^{-1}(c_0)$. Suppose that $\mu_1 = 0, \dots, \mu_r = 0$ are branches of divisors in $\phi^{-1}(c_0)$, and that

$\mu_{r+1} = 0, \ldots, \mu_n = 0$ are not. Then for $\mathrm{Re} s \gg 0$ $\oint\limits_{Y(\epsilon)} f|\lambda|^s|\omega|$ is the value at $s_{r+1} = \ldots = s_n = 0$ of $\int\limits_{Y(\epsilon)} f|\lambda|^s|\mu_{r+1}|^{s_{r+1}} \ldots |\mu_n|^{s_n}|\omega|$. Since $\lambda = \alpha\mu_1^{a_1} \ldots \mu_r^{a_r}$, where $|\alpha| \neq 0$ on the support of f, it follows that $\lim\limits_{\epsilon' \to 0} \oint\limits_{Y(\epsilon')} f|\lambda|^s|\omega| = 0$. Since ϕ is smooth away from the special fiber we have (cf. Lemma 1.F) that for $\epsilon' < \epsilon$ and $\mathrm{Re} s \gg 0$

$$\oint\limits_{Y(\epsilon)-Y(\epsilon')} f|\lambda|^s|\omega| = \int\limits_{I(\epsilon)-I(\epsilon')} F(\lambda)|\lambda|^s d\lambda \quad .$$

The asymptotic expansion for $F(\lambda)$ implies that

$$\lim_{\epsilon' \to 0} \int\limits_{I(\epsilon')} F(\lambda)|\lambda|^s d\lambda = 0 \text{ for } \mathrm{Re} s \gg 0 \quad .$$

Thus

$$\oint\limits_{Y(\epsilon)} f|\lambda|^s|\omega| = \int\limits_{I(\epsilon)} F(\lambda)|\lambda|^s d\lambda \quad .$$

Since the value of the left side at $s = 0$ is $\oint\limits_{Y(\epsilon)} f|\omega|$ the lemma is proved.

Lemma 2.C. *If $F_1(\theta, \beta, f) = 0$ for all $\beta \leq 0$ then*

$$\lim_{\epsilon \to 0} \oint\limits_{Y(\epsilon)} f|\omega| = 0 \quad .$$

Proof: Under this assumption the asymptotic expansion involves only *positive* exponents β. Then $\lim\limits_{\epsilon \to 0} \oint\limits_{I(\epsilon)} F(\lambda)d\lambda = 0$. Hence, by the last lemma, $\lim_{\epsilon \to 0} \oint\limits_{Y(\epsilon)} f|\omega| = 0$.

§3. **Some Igusa Data.**

Following [L, §§2-5] we now construct a smooth variety Y over F, an Igusa fibering $\phi : Y \longrightarrow C$ of Y over a curve C, a differential form ω of maximal degree on Y, and an integrand f_κ (notation of [L, §2]) on $Y(F)$.

Fix an inner form G of $SL(2)$ and a maximal torus T over F in G. Let c_0 be a point in the center of G. For the curve C we take T with the other central point removed.

The construction of Y starts with the variety S of stars ([L, §2]). Here S is just $\mathcal{B} \times \mathcal{B}, \mathcal{B}$ denoting the variety of Borel subgroups of G. Let $B_\infty \in \mathcal{B}$. Then $S(B_\infty)$ is $\mathcal{B} - \{B_\infty\} \times \mathcal{B} - \{B_\infty\}$. Let

$B_0 \in \mathcal{B} - \{B_\infty\}$. Then $S(B_\infty, B_0)$ consists of the pairs (B_+, B_-) in $S(B_\infty)$ with $B_+ = B_0$. If $N(\cdot)$ indicates unipotent radical and B^g the Borel subgroup $g^{-1}Bg, g \in \mathcal{B}$, we have

$$\begin{aligned} S(B_\infty) &= \{(B_0^{n_1}, B_0^{nn_1}); n, n_1 \in N(B_\infty)\} \\ &\simeq S(B_\infty, B_0) \times N(B_\infty) \\ &\simeq N(B_\infty) \times N(B_\infty) \ . \end{aligned}$$

Coordinates for $S(B_\infty)$ are evident, but the demands for Galois action require that a little care be taken in the choice.

First, and *for the rest of the paper*, we fix data as in [L, §2]: $G^* = SL(2)$, $\mathbf{B}^*$ is the upper triangular subgroup of G^*, $\mathbf{B}_*$ the lower triangular subgroup, $\mathbf{T}^*$ the diagonal subgroup; $\psi : G \longrightarrow G^*$ is an inner twist such that $\psi : T \longrightarrow G^*$ is defined over F, T^* denotes $\psi(T), \eta^* : G^* \longrightarrow G^*$ is a diagonalization of T^* and, finally, η denotes $\eta^* \circ \psi$.

By means of ψ we identify G with G^* as a group over $\bar{F}$, and hence $\mathcal{B}$ with $\mathcal{B}^*$ and S with $\mathcal{B}^* \times \mathcal{B}^*$, where $\mathcal{B}^*$ is the variety of Borel subgroups of $SL(2)$. View $\mathcal{B}^*$ as the variety $\mathbf{P}^1$ of lines through the origin in $\mathbf{A}^2$ via

$$(\mathbf{B}^*)^g \leftrightarrow [0,1] \cdot g \ .$$

Write a for $[a, 1]$.

Returning to B_∞ and B_0, now elements of $\mathcal{B}^*$, we choose $h \in G^*$ such that

$$(B_0)^h = \mathbf{B}^* \text{ and } (B_\infty)^h = \mathbf{B}_* \ . \tag{3.1}$$

Then h allows us to identify $S(B_\infty)$ with $S(\mathbf{B}_*)$. If $n = \left[\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right]$ and $n_1 = \left[\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right]$ we have

$$S(\mathbf{B}_*) \ni ((\mathbf{B}^*)^{n_1}, (\mathbf{B}^*)^{nn_1}) \leftrightarrow (y, x+y) \in \mathbf{P}^1 \times \mathbf{P}^1 \ . \tag{3.2}$$

Thus h provides coordinates, informally denoted x and y, on $S(B_\infty)$.

The variety $S_1(B_\infty)$ of [L, §3] is naturally identified with $S(B_\infty)$; S_1 is then obtained by gluing together the $S(B_\infty), B_\infty \in \mathcal{B}^*$, according to the rules of [L, (3.7) and (3.8)]. But these are the rules for the natural gluing of open subsets of $\mathcal{B}^* \times \mathcal{B}^* = \mathbf{P}^1 \times \mathbf{P}^1$, and so $S_1 = S = \mathbf{P}^1 \times \mathbf{P}^1$ (cf. [L, Lemma 3.10(a)]).

To describe the Galois action on S and at the same time maintain our identification of $G(\bar{F})$ with $G^*(\bar{F})$ and of S with $\mathcal{B}^* \times \mathcal{B}^*$ we equip $G^*(\bar{F})$ with the Galois action $\sigma_G = \psi \circ \sigma \circ \psi^{-1}, \sigma \in \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$.

Recall that the identification $\psi : T \longrightarrow T^*$ is over F. Let $L \subset \bar{F}$ be a quadratic extension of F. Write $\mathcal{T}_L$ for the set of tori in G defined over F (i.e., in G^* and preserved by $\sigma_G, \sigma \in \mathrm{Gal(\bar{F}/F)}$) which are anisotropic over F and split over L. We allow also $L = F$, then meaning by $\mathcal{T}_L$ the set of F-split tori in G.

The action of $\sigma \in \mathrm{Gal(\bar{F}/F)}$ on $S = \mathcal{B}^* \times \mathcal{B}^*$ will be denoted $\sigma_{(G,T)}$. From [L, §2 and §4] we get

$$\sigma_{(G,T)}((B_+, B_-)) = (\sigma_G(B_-), \sigma_G(B_+)) \tag{3.3}$$

if $T \in \mathcal{T}_L, L \neq F$, and σ is nontrivial on L, and

$$\sigma_{(G,T)}((B_+, B_-)) = (\sigma_G(B_+), \sigma_G(B_-)) \tag{3.4}$$

otherwise.

The following elaborate remark will be helpful later on.

(3.5) If $\sigma_G(\mathbf{B}^*) = \mathbf{B}^*$ and $\sigma_G(\mathbf{B}_*) = \mathbf{B}_*$, as we may assume if G is split over F, then S is covered by patches $S(B_\infty)$, where $\sigma_G(B_\infty) = B_\infty$ and $\sigma_G(B_0) = B_0, \sigma \in \mathrm{Gal(\bar{F}/F)}$, for some $B_0 \neq B_\infty$. For example, $S = S(\mathbf{B}_*) \cup S(\mathbf{B}^*)$. Each such patch $S(B_\infty)$ is preserved by $\mathrm{Gal(\bar{F}/F)}$. The element h of (3.1) can be chosen so that $\sigma_G(h)h^{-1}$ is central, $\sigma \in \mathrm{Gal(\bar{F}/F)}$. Then the identification of $S(B_\infty)$ with $S(\mathbf{B}_*)$ provided by h respects Galois action.

(3.6) Suppose that L is a quadratic extension of F. Assume, as we may if $T \in \mathcal{T}_L$, that $\sigma_G(\mathbf{B}^*) = \mathbf{B}_*$ for σ nontrivial on L and $\sigma_G(\mathbf{B}^*) = \mathbf{B}^*, \sigma_G(\mathbf{B}_*) = \mathbf{B}_*$ otherwise. Then S is covered by coordinate patches $S(B_\infty)$ where for some $B_0 \neq B_\infty, \sigma_G(B_\infty) = B_0$ for σ nontrivial on L and $\sigma_G(B_\infty) = B_\infty, \sigma_G(B_0) = B_0$ otherwise. Again $S = S(\mathbf{B}_*) \cup S(\mathbf{B}^*)$ will do. Now, however, $\sigma_{(G,T)}$ preserves only $S(B_\infty) \cap S(B_0) = S(B_\infty) - \{(B_0, B_0)\}$ if σ is nontrivial on L. The element h of (3.1) may be chosen so that $\sigma_G(h)h^{-1}$ is central, $\sigma \in \mathrm{Gal(\bar{F}/F)}$. Then the identification of $S(B_\infty) \cap S(B_0)$ with $S(\mathbf{B}_*) \cap S(\mathbf{B}^*)$ provided by h respects Galois action.

Returning to the construction of Y we find it convenient to make yet another identification, that of T and T^* with $\mathbf{T}^*$ using the diagonalization η^*. We equip $\mathbf{T}^*(\bar{F})$ with the action $\sigma_T = \sigma_{T^*} = \eta^* \circ \sigma \circ (\eta^*)^{-1}|_{\mathbf{T}^*}, \sigma \in \mathrm{Gal(\bar{F}/F)}$, and regard C as a curve in $\mathbf{T}^*$ preserved by this action. Note that $\eta^*(c_0) = c_0$.

A star $s = (B_+, B_-)$ is regular in the sense of [L, §2] if and only if $B_+ \neq B_-$. For the variety X_1 of [L, §4] we take the closure in $G^* \times S$ of $\{(g, s = (B_+, B_-)) : g, s \text{ regular}, g \in B_+ \cap B_-\}$; X_1 is defined over F for the Galois action given by $\sigma_G \times \sigma_{(G,T)}, \sigma \in \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$. There are maps defined over F:

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi} & G^* \\ \Big\downarrow{\scriptstyle \phi_1} & & \\ \mathbf{T}^* & & \end{array}$$

where $\sigma \in \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$ acts on G^* by σ_G and on $\mathbf{T}^*$ by σ_T. The horizontal arrow is projection on the first component. To define ϕ_1, note that X_1 is contained in $\{(g, s = (B_+, B_-)) \in G^* \times S : g \in B_+ \cap B_-\}$. Thus if $(g, s) \in X_1$ we may choose $h \in G^*$ such that $B_+^h = \mathbf{B}^*$. Then $\phi_1((g, s))$ is the image of $h^{-1}gh \in \mathbf{B}^*$ under the projection $\mathbf{B}^* = \mathbf{T}^* N(\mathbf{B}^*) \longrightarrow \mathbf{T}^*$.

The variety Y will be the intersection of $\phi_1^{-1}(C)$ with the closure in X_1 of $\phi_1^{-1}(C - \{c_0\})$. By restriction we have:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & G^* \\ \Big\downarrow{\scriptstyle \phi} & & \\ C & & \end{array}$$

Let M be the Springer-Grothendieck variety $\{(g, b) : g \in B\} \subset G^* \times \mathcal{B}^*$, with the usual maps:

$$\begin{array}{ccc} M & \xrightarrow{\pi_M} & G^* \\ {\scriptstyle \phi_M}\Big\downarrow & & \\ \mathbf{T}^* & & \end{array}$$

Define $\xi : Y \longrightarrow M$ by $(g, (B_+, B_-)) \longrightarrow (g, B_+)$. Then $\phi = \phi_M \circ \xi$ and $\pi = \pi_M \circ \xi$. If M' is M with the fibers over the central points removed then $\phi^{-1}(C - \{c_0\}) \xrightarrow{\xi} M'$ is an isomorphism of varieties over $\bar{F}$. In particular, $\phi^{-1}(C - \{c_0\})$ is smooth.

To examine the special fiber $\phi^{-1}(c_0)$ we introduce coordinates as in [L, §3]. Let $Y \subset G^* \times S \longrightarrow S$ be projection on the second factor. Let $Y(B_\infty)$ be the inverse image of $S(B_\infty), B_\infty \in \mathcal{B}^*$. Identify $S(B_\infty)$ with $S(\mathbf{B}_*)$ by means of some h as in (3.1). We may then work with the coordinates x, y of (3.2) on $S(\mathbf{B}_*)$ and with $Y(\mathbf{B}_*)$.

Let λ be a local F-coordinate around c_0 in C. Recall that $C \subset \mathbf{T}^*$ and that $\sigma \in \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$ acts by σ_T. Assume that $\lambda = 0$ at c_0. If α is the root of $\mathbf{T}^*$ in $\mathbf{B}^*$ then we may write $1 - \alpha^{-1}$ as $\lambda b(\lambda)$ near c_0, with b regular and invertible near $\lambda = 0$. Suppose that $(g, s) \in Y(\mathbf{B}_*)$. As in (3.2) write s as

$((\mathbf{B}^*)^{n_1}, (\mathbf{B}^*)^{nn_1})$, with $n = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ and $n_1 = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$. Note that x is the coordinate $z(W_+, \alpha)$ from [L, §3]. Write

$$g = n_1^{-1} t \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} n_1, \text{ with } t \in \mathbf{T}^*, u \in \bar{F} \ . \tag{3.7}$$

Assume $x \neq 0$. Then $g \in (B^*)^{nn_1}$ is equivalent to

$$1 - \alpha(t)^{-1} = xu$$

or, if (g, s) is near $\phi^{-1}(c_0)$ and we pull back λ to Y, to

$$\lambda b(\lambda) = xu \ . \tag{3.8}$$

As a consequence, u, x and y serve as coordinates on $Y(\mathbf{B}_*)$, and $Y(\mathbf{B}_*)$ is smooth. Then each $Y(B_\infty)$ is smooth, $B_\infty \in \mathcal{B}^*$. Hence Y is a smooth variety.

Near $\phi^{-1}(c_0)$ on $Y(\mathbf{B}_*)$, ϕ is given by

$$\lambda = Axu \tag{3.9}$$

with A regular and invertible near $\lambda = 0$. Thus $u = 0$ is the branch of a divisor E_1 of $\phi^{-1}(c_0)$. This branch consists of the pairs $(c_0, s), s \in S(\mathbf{B}_*)$, and so E_1 must be $\{c_0\} \times S = \{c_0\} \times \mathbf{P}^1 \times \mathbf{P}^1; E_1$ maps under $\pi : Y \longrightarrow G$ to $\{c_0\}$. On the other hand, $x = 0$ is the branch $\{(g, (B, B)) : B \neq B_*, g \in G, c_0 g \text{ unipotent}\}$. For convenience we call g c_0-unipotent if $c_0 g$ is unipotent. Then π maps $E_2 - E_1$ isomorphically to the orbit of regular c_0-unipotent elements in G. Note that the two divisors E_1 and E_2 cover $\phi^{-1}(c_0)$, and that E_2 has no F-rational points unless G is split over F.

The relation of Y to the Springer-Grothendieck variety M is now evident. Under $Y \xrightarrow{\xi} M$ the divisor E_2 is mapped isomorphically to the fiber over c_0; Y is obtained from M with the fiber over $-c_0$ removed by blowing up along the subvariety $\{c_0\} \times \mathcal{B}^* = \xi(E_1)$ of the fiber over c_0.

To verify that $\phi : Y \longrightarrow C$ is an Igusa fibering it remains only to check that one of E_1, E_2 is defined over F. For then both divisors are defined over F and we may apply (3.9) and Hilbert's Theorem 90 (for the field of functions regular and invertible near a point) to replace around each F-rational point $y_0 \in \phi^{-1}(c_0)$ the coordinates u, x and y with F-coordinates μ_1, μ_2, μ_3 such that:

(3.10) $\mu_i = 0$ is a branch of E_i if y_0 lies on $E_i (i = 1, 2)$.

(3.11) λ is given near y_0 by $\lambda = \alpha\mu_1^{a_1}\mu_2^{a_2}$, where α is regular and invertible at y_0, and $a_i = 1$ if y_0 lies on E_i and $a_i = 0$ otherwise $(i = 1, 2)$.

Since $E_1 = \{c_0\} \times S$ is clearly defined over F, we are done.

The indices $a(\cdot)$ of §2 are:

$$\text{(3.12)} \qquad \begin{aligned} &a(E_1) = a(E_2) = 1 \text{ if } G \text{ is split over } F\ , \\ &a(E_1) = 1 \qquad\qquad \text{otherwise.} \end{aligned}$$

The next step is to define an Igusa form ω. Let ω_T be the (right) invariant form on $\mathbf{T}^*$ equal to $d\lambda$ at c_0. Let $a \in \bar{F}$ be such that $\bar{\omega} = a\omega_T$ is defined over F for the Galois action on $\mathbf{T}^*$ as F-split torus. Let $H \in \mathrm{Lie}(\mathbf{T}^*)$ be such that $\bar{\omega}(H) = 1$. Choose $X_+ \in \mathrm{Lie}(N(\mathbf{B}^*)), X_- \in \mathrm{Lie}(N(\mathbf{B}_*))$ and right invariant 1-forms $\omega, \omega_+, \omega_-$ on G defined over F so that $\langle\omega_0, \omega_+, \omega_-\rangle$ is dual to $\langle H, X_+, X_-\rangle$. Then $\omega_G = \omega_0 \wedge \omega_+ \wedge \omega_-$ is a (right) invariant form of maximal degree on G defined over F. The form ω_M on M associated to ω_G (more precisely, to $\nu_1 = \omega_0, \nu_2 = \omega_+, \omega_1 = \omega_-$) in [L, Lemma 2.8] is $\pi_M^*(\omega_G)$. We set $\omega_Y = \xi^*(\omega_M) = \pi^*(\omega_G)$ and $\omega = a^{-1}\omega_Y$.

The form ω is regular; it is nonvanishing off the special fiber. The discussion of [L, §2] implies that locally $\omega = W'\omega'$, where W' is a regular invertible function and ω' is defined over F. This ensures that the measure $|\omega|$ is well-defined.

Suppose that $y_0 \in Y(\mathbf{B}_*)$ is near but not on $\phi^{-1}(c_0)$. We may as well take $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then it may be shown that ω is given near y_0 by $\phi^*(d\lambda) \wedge du \wedge dy = W(\lambda) d(xu) \wedge du \wedge dy = W(\lambda) u\ dx\ \wedge du \wedge dy$, where W is regular and invertible near $\lambda = 0$, with $W(0) = (\mathrm{ab}(0))^{-1}$ (cf. (3.8)). From this it follows that

$$\text{(3.13)} \qquad \omega = W(\lambda) u^2 x \frac{dx}{x} \wedge \frac{du}{u} \wedge dy$$

around a point of $Y(\mathbf{B}_*) \cap \phi^{-1}(c_0)$.

Note that ω may be expressed in terms of the coordinates μ_1, μ_2, μ_3 of (3.10), but that the coordinates u and x will do just as well to compute the indices $b(\cdot)$ of §2:

$$\text{(3.14)} \qquad \begin{aligned} &b(E_1) = 2, \quad b(E_2) = 1 \text{ if } G \text{ is split over } F \\ &b(E_1) = 1 \qquad\qquad \text{otherwise.} \end{aligned}$$

It remains to define the Igusa integrand. Let κ be a character on $\mathcal{D}(T) = \mathcal{D}(T, F)$, the definition of which will be recalled in (3.15). Recall that $T(\bar{F})\backslash\mathfrak{A}(T, F)$ is the set of F-rational points in $T(\bar{F})\backslash G(\bar{F}) = (T\backslash G)(\bar{F})$. If $\gamma \in T(F) - \{c_0\}$ then $\pi : Y \longrightarrow G$ induces an F-isomorphism from the fiber ϕ_γ^{-1} over γ in Y to $T\backslash G$ (cf. [L, Lemma 2.1]). We have therefore:

$$\begin{aligned}(3.15)\qquad \phi_\gamma^{-1}(F) \longrightarrow T(\bar{F})\backslash\mathfrak{A}(T, F) \longrightarrow \mathcal{D}(T, F) &= T(\bar{F})\backslash\mathfrak{A}(T, F)/G(F)\\ &\simeq H^1(\mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F}), \mathrm{T}(\bar{\mathrm{F}}))\ ,\end{aligned}$$

allowing us to regard κ as a function m_κ on $\phi_\gamma^{-1}(F)$. If κ is trivial then $m_\kappa \equiv 1$. Suppose then that $T \in \mathcal{T}_L, L \neq F$, and κ is nontrivial. We will need an explicit formula for m_κ near an F-rational point y_0 on the special fiber. Proposition 5.1 of [L] shows that m_κ depends locally only on the coordinate x, at least if G is split over F, but for the formula we will need an F-coordinate.

Suppose that $S(B_\infty)$ is a coordinate patch as in (3.5). We may as well take $\sigma_G = \sigma_{G^*}, \sigma \in \mathrm{Gal}(\bar{\mathrm{F}}/\mathrm{F})$, or $G = SL(2)$. Identify $S(B_\infty)$ with $S(\mathbf{B}_*)$ using h as in (3.5). Recall that this identification respects the Galois action on S. The formulas (3.2), (3.3) and (3.4) imply that the coordinates x, y on $S(\mathbf{B}_*)$ satisfy $\sigma(y) = x + y$ if σ is nontrivial on L and that $\sigma(y) = y, \sigma(x + y) = x + y$ otherwise. Then $\sigma(x) = \sigma(x + y - y) = y - (x + y) = -x$ for σ nontrivial on L. Fix $\tau \in L - F$ such that $\tau^2 \in F$. Then $\mu = \tau x$ is an F-coordinate (and will serve as μ_2 in (3.10)). Let $(g, s) \in \phi_\gamma^{-1}(F)$ lie in $Y(B_\infty)$, which we have identified with $Y(\mathbf{B}_*)$. The coordinate μ then being F-valued, we have that

$$\sigma \longrightarrow \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \text{ if } \sigma|_L \not\equiv 1, \sigma \longrightarrow 1 \text{ otherwise}\ ,$$

represents an element of $H^1(T)$ that we denote μ_σ. Let ϵ_σ denote the image of (g, s) under (3.15). Then Proposition 5.2 of [L] implies that there is an element t_σ of $H^1(T)$ independent of (g, s) such that

$$(3.16)\qquad \epsilon_\sigma = \mu_\sigma t_\sigma$$

(see the Appendix to this section). Thus

$$m_\kappa((g, s)) = \kappa(\epsilon_\sigma) = \kappa(\mu_\sigma)\kappa(t_\sigma) = \kappa(\mu)\kappa(t_\sigma)\ ,$$

where κ now also denotes the quadratic character on $F^\times$ attached to L/F.

By requiring that $S(B_\infty)$ be as in (3.5) we have excluded the case G anisotropic over F. This is of no consequence, for then if $(g, s) \in Y(F)$ the star $s = (B_+, B_-)$ must be regular $(\sigma_G(B_+ \cap B_-) = B_+ \cap B_-$

implies $B_+ \cap B_-$ is a torus so that $B_+ \neq B_-$). From Lemma 2.10 of [L] we conclude that m_κ is locally constant on $Y(F)$.

Finally, fix $f \in C_c^\infty(G(F))$. The Igusa integrand will be:

$$\begin{aligned} f_\kappa(g,s)) &= m_\kappa((g,s))(f \circ \pi)(g,s) \\ &= m_\kappa((g,s))f(g), \qquad\qquad (g,s) \in Y(F) - \phi^{-1}(c_0)\ . \end{aligned}$$

The characters $\kappa(\cdot)$ of §2 are:

(3.17)
$$\begin{cases} \kappa(E_1) \equiv 1 \text{ and } \kappa(E_2) = \kappa & \text{if G is split over F} \\ \kappa(E_1) \equiv 1 & \text{otherwise.} \end{cases}$$

Appendix

Here we note the explicit calculation of ϵ_σ in (3.16) and another local expression for m_κ which applies to anisotropic groups as well.

For (3.16) recall that we have assumed that G is $SL(2)$ (and $\psi \equiv 1, T = T^*$). We refrain from identifying $T(\bar{F})$ with $\mathbf{T}^*(\bar{F})$. Then ϵ_σ is the class of $\sigma \longrightarrow \sigma(h_1)h_1^{-1}$, where $h_1 \in G(L)$ satisfies:

$$h_1 g h_1^{-1} \in T, B_+^{h_1} = \eta^{-1}(\mathbf{B}^*) \text{ and } B_-^{h_1} = \eta^{-1}(\mathbf{B}_*)$$

if $s = (B_+, B_-)$. Write η as $t \longrightarrow h_2 t h_2^{-1}, h_2 \in G(L)$. For h_1, we can take $h_2^{-1}h_3$ if $h_3 \in G(L)$ satisfies:

$$h_3 g h_3^{-1} \in \mathbf{T}^*, B_+^{h_3} = \mathbf{B}^* \text{ and } B_-^{h_3} = \mathbf{B}_*\ .$$

On $Y(\mathbf{B}_*)$ we have

$$g = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix} t \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix},$$

with $t \in \mathbf{T}^*$ and $ux = 1 - \alpha(t)^{-1}$. It is easily checked that

$$h_3 = \begin{pmatrix} 1 & 1/x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

will do;

$$\sigma(h_3)h_3^{-1} = \begin{pmatrix} 0 & -1/x \\ x & 0 \end{pmatrix} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 0 & -1/\tau \\ \tau & 0 \end{pmatrix}$$

for σ nontrivial on L. Then

$$\begin{aligned} \sigma(h_1)h_1^{-1} &= \sigma(h_2^{-1}) \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 0 & -1/\tau \\ \tau & 0 \end{pmatrix} h_2 \\ &= \eta^{-1} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \sigma(h_2^{-1}) \begin{pmatrix} 0 & -1/\tau \\ \tau & 0 \end{pmatrix} h_2\ , \end{aligned}$$

so that (3.16) holds with τ_σ the class of

$$\sigma \longrightarrow \sigma(h_2^{-1}) \begin{pmatrix} 0 & -1/\tau \\ \tau & 0 \end{pmatrix} h_2 \ .$$

Suppose now that L is a quadratic extension of F and that

$$\sigma_G = \begin{cases} \operatorname{ad}\begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} \circ \sigma_{G^*} & \text{if } \sigma|_L \not\equiv 1 \ , \\ \sigma_{G^*} & \text{otherwise} \ , \end{cases} \tag{3.18}$$

where $\zeta \in F^\times$. Note that G is split over F if and only if $\zeta \in \mathrm{Nm}_F^L L^\times$. Assume that $T \in \mathcal{T}_L$. Now σ_G satisfies the conditions of (3.6). The coordinates x and y on $S(\mathbf{B}_*)$ satisfy $\sigma(x+y) = \zeta/y, \sigma(y) = \zeta/(x+y)$ if $\sigma|_L \not\equiv 1$ and $\sigma(x+y) = x+y, \sigma(y) = y$ otherwise. Then $x/(x+y) = 1 - y/(x+y)$ is defined over F. It serves as a coordinate around an F-rational point (g,s) of $Y(\mathbf{B}_*)$ near $\phi^{-1}(c_0)$. A calculation as in the last paragraph shows that:

$$m_\kappa((g,s)) = \kappa(x/(x+y)) \ , \tag{3.19}$$

where κ now denotes the quadratic character of $F^\times$ attached to L/F if κ is nontrivial, and the trivial character otherwise.

§4. Application

Continuing from the last section, we have $f \in C_c^\infty(G(F))$, Haar measures $|\omega_G|$ on $G(F)$ and $|\omega_T|$ on $T(F)$, and a character κ on $\mathcal{D}(T)$. For γ regular in $T(F)$, form the κ-orbital integral

$$\begin{aligned} \Phi^\kappa(\gamma, f) &= \Phi^\kappa_T(\gamma, f, |\omega_T|, |\omega_G|) \\ &= \sum_\delta \kappa(\delta) \int\limits_{T^h(F)\backslash G(F)} f(g^{-1}h^{-1}\gamma h g) \frac{|\omega_G|}{|\omega_T|^h}, \end{aligned}$$

where $h \in \mathfrak{A}(T,F)$ represents $\delta \in \mathcal{D}(T,F)$, and then the normalized integral

$$F^\kappa(\gamma, f) = |1 - \alpha(\gamma^{-1})| \Phi^\kappa(\gamma, f).$$

Recall that T has been identified with $\mathbf{T}^*$ by means of η; α is the root of $\mathbf{T}^*$ in $\mathbf{B}^*$.

Assume that γ lies in $C(F)$ near c_0 and has coordinate λ. The Igusa data of the last section were chosen so that

$$F^\kappa(\gamma, f) = \int\limits_{\phi_\gamma^{-1}(F)} f_\kappa \frac{|\omega|}{|d\lambda|}.$$

Thus for $|\lambda|$ sufficiently small we have that

$$F^\kappa(\gamma, f) = |\lambda|\Lambda_1 + \kappa(\lambda)\Lambda_2 \tag{4.1}$$

where, in the notation of Lemma 2A,

$$\Lambda_1 = F_1(1, 2, f) \tag{4.2}$$

and

$$\Lambda_2 = \begin{cases} F_1(\kappa, 1, f) & \text{if } G \text{ is split over } F \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

On the right side of (4.1) we have, as in §3, regarded κ as a character on $F^\times$, trivial if κ is trivial on $\mathcal{D}(T)$ and the character on $F^\times$ attached to the quadratic splitting field L of T otherwise.

The term Λ_1 is the contribution from the divisor E_1 which maps to $\{c_0\}$ under π: $Y \to G$, while Λ_2 is the contribution from E_2; under π, E_2 maps (isomorphically) to the conjugacy class of regular c_0-unipotent elements in G. Thus (4.1) assumes a familiar form, but with the coefficients now expressed as principal value intregrals.

If T is split over F then blowing up the Springer-Grothendieck variety M to obtain Y is unnecessary, and as a consequence we have introduced the spurious term Λ_1. It is quickly dismissed, for if T is split over F then $E_1 = \{c_0\} \times S$ is F-isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ (cf. (3.4)) and by (2.10) Λ_1 is given, up to a constant, by

$$\oint\limits_{\mathbf{P}^1(F)\times\mathbf{P}^1(F)} \frac{da\,db}{|a-b|^2}$$

where a, b each denote the coordinate U_0 on $U_1 = 1$ in $\mathbf{P}^1$. We apply Lemma 1.F to this integral and the fibering ϕ: $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ given by projection on the first component. The fiber integral $H(x')$, $x' \in \mathbf{P}^1(F)$, is seen to be an integal over $\mathbf{P}^1(F)$ of the form (1.15). Thus it is zero. We conclude then from Lemma 1.F that $\Lambda_1 = 0$.

Lemma 4.A. *If κ is nontrivial then $\Lambda_1 = 0$.*

Proof: If G is anisotropic over F then this is immediate from the definition of κ-orbital integral (see also the remark following the proof of Lemma 4.B). Suppose then that G is split over F. We may as well assume that $G = G^* = \mathrm{SL}(2)$. Since κ is nontrivial $T \in \mathcal{T}_L$, some $L \neq F$; E_1 is the variety $R = \mathrm{Res}_{\mathrm{F}}^{\mathrm{L}}\mathbf{P}^1$ of Remark 1.H, i.e., $E_1(L) = \mathbf{P}^1(L) \times \mathbf{P}^1(L)$ and $E_1(F) = \{(p, \bar{p})\colon p \in \mathbf{P}^1(L)\}$, where the bar denotes the action of the nontrivial element of $\mathrm{Gal}(L/F)$ (cf. (3.3), (3.4)). Then by (2.10), (3.2), (3.5) and (3.16) λ_1 is, up to a constant,

$$\oint_{R(F)} \frac{\kappa\left(\frac{b-a}{\tau}\right) da\, db}{|b-a|^2} \tag{4.4}$$

where a, b each denote the coordinate U_0 on $U_1 = 1$ in $\mathbf{P}^1(L)$. The element τ of $L - F$ was fixed for (3.16); $\frac{b-a}{\tau}$ lies in $F^\times$ if $b = \bar{a} \neq 0$.

Abbreviate the point $U_0 = a$, $U_1 = 1$ in $\mathbf{P}^1$ by a, and $U_0 \neq 0$, $U_1 = 0$ by ∞. We define a smooth morphism ϕ: $R - \{(\infty,\infty)\} \to \mathbf{P}^1$ by $(a,b) \to \frac{b-a}{\tau}$ and $(a,\infty), (\infty,b) \to \infty$; ϕ^0 is defined over F. We blow up R at (∞,∞) to obtain the variety $\widehat{R}$ over F. The fiber over (∞,∞) in $\widehat{R}$ meets the proper inverse image of the divisor $a = b$ at a single point p_0. Blow up $\widehat{R}$ at this point, which is F-rational, to obtain the variety $\widehat{\widehat{R}}$ over F. A calculation with coordinates shows that ϕ^0 extends to an Igusa fibering ϕ: $\widehat{\widehat{R}} \to \mathbf{P}^1$ with distinguished point $c_0 = \infty$; moreover, the fiber over ∞ is the union of three divisors, each occurring with multiplicity one. Only one of the divisors has F-rational points. We conclude then that on $\widehat{\widehat{R}}(F)$ the map ϕ is smooth.

To compute (4.4) by lifting to $\widehat{\widehat{R}}(F)$ we must check the conditions of Lemma 1.D at (∞,∞) on $R(F)$ and at p_0 on $\widehat{R}(F)$. We find that $\theta_1 = \kappa$, $c_1 = -1$ and $\theta_2 \equiv 1$, $c_2 = 1$ for a suitable choice of local coordinates around (∞,∞) on $R(F)$. Thus $\theta_1\theta_2 = \kappa$ and $c_1 + c_2 = 0$, and so to lift to $\widehat{R}(F)$ it is crucial that κ be nontrivial. On $\widehat{R}(F)$ at p_0 we find $\theta_1 = \kappa$, $c_1 = 0$ and $\theta_2 = \kappa$, $c_2 = -1$. The conditions of Lemma 1.D are met and we may apply Corollary 1.E to rewrite (4.4) as a principal value integral I over $\widehat{\widehat{R}}(F)$. We compute I by applying Lemma 1.F to the fibering ϕ. For any $p \in \mathbf{P}^1(F) - \{0,\infty\}$ the corresponding fiber integral is seen immediately to be a constant times $\oint_{\mathbf{P}^1(F)} da = 0$ (cf. (1.15)). Thus Lemma 1.F implies that $I = 0$, and Lemma 4.A is proved.

Suppose now that $T \in \mathcal{T}_L$, $L \neq F$; κ may be either character on $\mathcal{D}(T)$. Recall that the maximal torus $T^* = \psi(T)$ in G^* is defined over F, as is the map ψ: $T \to T^*$. Let κ^* be the character on $\mathcal{D}(T^*)$ associated to κ by ψ. We indicate by Λ_1^* the contribution (4.2) for the data G^*, T^*, κ^* and $f^* \in C_c^\infty(G^*(F))$. It may be written as $M_1^* f^*(c_0)$. Similarly the contribution (4.2) for the data G, T, κ and f can be written as $M_1 f(c_0)$.

Lemma 4.B.

$$M_1 = \varepsilon(\kappa, G) M_1^*$$

where $\varepsilon(\kappa, G) = 1$ *if* κ *is nontrivial and*

$$\varepsilon(1, G) = \begin{cases} 1 & \textit{if } G \textit{ is split over } F \\ -1 & \textit{otherwise.} \end{cases}$$

Proof: We may assume that G satisfies (3.18), i.e.,

$$\sigma_G = \begin{cases} \operatorname{ad}\begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} \circ \sigma_{G^*} & \text{if } \sigma\big|_L \not\equiv 1 \\ \sigma_{G^*} & \text{otherwise,} \end{cases}$$

where $\zeta \in F^\times$; G is split over F if and only if $\zeta \in \mathrm{Nm}_F^L L^\times$. We write $M_1(\zeta)$ for the term M_1. Then from (2.10), (3.2), (3.6) and (3.19) we find that $M_1(\zeta)$ is given up to a constant independent of G (i.e., of ζ) by

$$\oint_{Q_\zeta(F)} \frac{\kappa\left(1 - \frac{1}{b}\right) da\, db}{|b-a|^2} \tag{4.5}$$

where Q_ζ is the form of $\mathbf{P}^1 \times \mathbf{P}^1$ on which $1 \neq \sigma \in \mathrm{Gal}(L/F)$ acts on the homogeneous coordinates U_0, U_1 (on the first copy of $\mathbf{P}^1$) and V_0, V_1 (on the second copy) by $U_0 \to \zeta V_1$, $U_1 \to V_0$, $V_0 \to \zeta U_1$, $V_1 \to U_0$. Also a denotes U_0 on $U_1 = 1$ and $b = V_0$ on $V_1 = 1$.

We define now a smooth variety Y and an Igusa fibering ϕ: $Y \to \mathbf{A}^1$ with distinguished point zero on $\mathbf{A}^1$ such that if ζ is the coordinate on $\mathbf{A}^1$ then (4.5) is the fiber integral $F(\zeta)$, $\zeta \neq 0$. The asymptotic expansion for $F(\zeta)$ at $\zeta = 0$ will be seen to have the one term, that corresponding to $\theta = \theta_L \kappa$, where θ_L is the quadratic character of $F^\times$ attached to L/F, and $\beta = 1$. Then in the notation of Lemma 2.A we have

$$F(\zeta) = \theta_L(\zeta)\kappa(\zeta) F_1(\theta_L \kappa, 1, ^*) \tag{4.6}$$

for $|\zeta|$ sufficiently small, where * is the Igusa integrand yet to be defined. This will prove the lemma.

We start with a variety $Y_1 \subset (\mathbf{P}^1)^4 \times \mathbf{A}^1$. let a, b, a_1, b_1 each denote the coordinate U_0 on $U_1 = 1$ in $\mathbf{P}^1$, and ζ be the coordinate on $\mathbf{A}^1$. On $(U_1 = 1)^4 \times \mathbf{A}^1$, Y_1 is given by $ab_1 = a_1 b = \zeta$. Let a', b', a_1', b_1' each denote U_1 on $U_0 = 1$. On $(U_0 = 1) \times (U_1 = 1)^3 \times \mathbf{A}^1$, Y_1 is given by $a_1 b = \zeta$ and $b_1 = a'\zeta$; on $(U_0 = 1)^2 \times (U_1 = 1)^2 \times \mathbf{A}^1$ by $b_1 = a'\zeta$, $a_1 = b'\zeta$, and so on. We define Y_1 over F by twisting the natural F-structure by $\sigma(a)=a_1$, $\sigma(b)=b_1$, $\sigma(a')=a_1'$, $\sigma(b')=b_1'$, σ being the nontrivial element of $\mathrm{Gal}(L/F)$.

The variety Y_1 is smooth except at the point y_1 given by $a=a_1=b=b_1=0$. Let ϕ_1 be the projection of $Y_1\subset(\mathbf{P}^1)^4\times\mathbf{A}^1$ onto $\mathbf{A}^1$; ϕ_1: $Y_1-\{y_1\}\to\mathbf{A}^1$ is an Igusa fibering with distinguished point zero.

For $\zeta\neq 0$ the projection of Y_1 onto the product of the first and second copies of $\mathbf{P}^1$ yields an F-isomorphism of the fiber $\phi_1^{-1}(\zeta)$ with Q_ζ. Let $Y_1'=Y_1-\phi_1^{-1}(0)$. We define a form ω_1 on Y_1' by $\frac{da\wedge db\wedge d\zeta}{(b-a)^2}$ and an integrand $f_1=\kappa\left(1-\frac{a}{b}\right)$ on $Y_1'(F)$. The fiber integral

$$\oint\limits_{\phi_1^{-1}(\zeta)(F)} f_1\,\frac{|\omega_1|}{|d\zeta|} \tag{4.7}$$

is (4.5).

We now attend to the fiber in $Y_1-\{y_1\}$ over $\zeta=0$. It is the union of divisors $E_1',\dots,E_4'$. Their branches on $((U_1=1)^4\times\mathbf{A}^1)\cap(Y_1-\{y_1\})$ are:

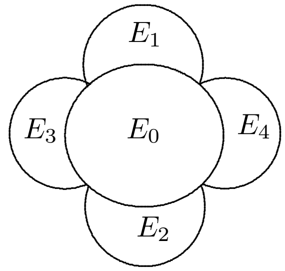
$$(E_1')\quad b=0,\, b_1=0 \qquad (E_2')\quad a=0,\, a_1=0$$

$$(E_3')\quad a=0,\, b=0 \qquad (E_4')\quad a_1=0,\, b_1=0.$$

Note that E_1', E_4' have no F-rational points; E_1', E_2' are each defined over F.

The point y_1 on Y_1 is F-rational. Blow up Y_1 at this point to obtain the variety Y over F and projection π: $Y\to Y_1$. Set $\phi=\phi_1\circ\pi$, $\omega=\pi^*(\omega_1)$ and $f_Y=f_1\circ\pi$. Then $Y, C=\mathbf{A}^1$, $c_0=0$, ϕ,ω and f_Y satisfy the conditions of (2.1)–(2.7) (i.e., are "generalized" Igusa data), as well as (2.9). The proof is routine. We will include as much of it as will be needed to write down the asymptotic expansion for the fiber integral which, by construction, coincides with the integral (4.7).

The fiber $\phi^{-1}(0)$ is the union of five divisors $E_0, E_1,\dots,E_4$, where E_i is the proper inverse image of E_i' $(i=1,\dots,4)$. Let $u_1=a$, $u_2=b$, $u_3=a_1$, $u_4=b_1$; let $U_1=A$, $U_2=B$, $U_3=A_1$, $U_4=B_1$ be homogeneous coordinates on $\mathbf{P}^3$. Then Y is given near $\pi^{-1}(y_1)$ by $u_iU_j=u_jU_i$ $(i,j=1,\dots,r)$. The divisor E_0 is given by $a=b=a_1=b_1=0$ and $AB_1=BA_1$ (homogeneous coordinates); on $E_0\cap E_1$ we have $B=0$, $B_1=0$; on $E_0\cap E_2$, $A=0$ and $A_1=0$, and so on. The divisors E_0, E_1 and E_2 are each defined over F, while E_3 and E_4 have no F-rational points and so may be ignored for the asymptotic expansion. Also $E_1 \cap E_2$ is empty and $E_i \cap E_j$ ($i = 1, 2$, and $j = 3, 4$) consists of a single point on E_0 which is not F-rational.



The variety E_0 is a form of $\mathbf{P}^1 \times \mathbf{P}^1$, the natural projections being given by $(A, B, A_1, B_1) \to \frac{B}{A} = \frac{B_1}{A_1}, \frac{A_1}{A} = \frac{B_1}{B}$ (where we allow the value ∞ and ignore quotients of the form $\frac{0}{0}$). For these to be defined over F, the first $\mathbf{P}^1$ has to be provided with its natural F-structure and the second with the structure of Remark 1.G. The variety E_1 is the blow-up $\bar{R}$ of the twisted form R of $\mathbf{P}^1 \times \mathbf{P}^1$ described in Remark 1.H (cf. also (4.4)). To see this, we note that the projection of $Y_1 \subset (\mathbf{P}^1)^4 \times \mathbf{A}^1$ onto the product of the first and third copies of $\mathbf{P}^1$ yields an F-isomorphism of E_1' with $R - \{r_0\}$, where r_0 is given by $a = a_1 = 0$. Then E_1 is the blow-up $\bar{R}$ of R at r_0, and $E_0 \cap E_1$ is the inverse image of r_0 in $\bar{R}$. The divisor E_2 is described similarly.

Suppose that $y_0 \in E_0(F)$ and that $A \neq 0$ at y_0. We may assume $A = 1$. Then $t = a_1$, A_1, B serve as coordinates on Y near y_0; $a_1 = tA_1$ and $b = tB$. For $1 \neq \sigma \in \mathrm{Gal}(L/F)$ we have $\sigma(B) = B$, $A_1\sigma(A_1) = 1$ and $\sigma(t) = tA_1$. To obtain F-coordinates we may take B, r and s with $t = st_0$, $t_0 \neq 0$, $\sigma(t_0)/t_0 = A_1$ and $t_0 = 1 + \tau r$, where $\tau \in L - F$ and $\tau^2 \in F$. Also, $t = 0$ is a branch of E_0, $A_1 \neq 0$ since $A_1\sigma(A_1) = 1$, and $B = 0$ is a branch of E_1. Finally, $\zeta = a_1 b = t^2 A_1 B$,

$$\omega = \frac{da \wedge db \wedge d\zeta}{(b-a)^2} = tB\frac{dt \wedge dB \wedge dA_1}{(B-1)^2}$$

and $f_Y = \kappa(1 - 1/B) = \kappa(B)\kappa(B-1)$. We conclude that

$$\beta(E_0) = \frac{b(E_0)}{a(E_0)} = \frac{2}{2} = 1,\ \kappa(E_0) \equiv 1.$$

Similarly,

$$\beta(E_1) = \frac{2}{1} = 2,\ \kappa(E_1) = \kappa$$

and

$$\beta(E_2) = \frac{2}{1} = 2,\ \kappa(E_2) \equiv 1.$$

This implies that (2.9) is satisfied (i.e., $e(\theta,\beta)\le 1$ for all (θ,β)) since E_1 and E_2 do not intersect.

The asymptotic expansion for the fiber integral, i.e., for the integral (4.5), is then

$$\sum_\theta \theta(\zeta)F_1(\theta,1,f_Y)+|\zeta|\kappa(\zeta)F_1(\kappa,2,f_Y)+|\zeta|F_1(1,2,f_Y) \tag{4.8}$$

if κ is nontrivial, or

$$\sum_\theta \theta(\zeta)F_1(\theta,1,f_Y)+|\zeta|F_1(1,2,f_Y) \tag{4.9}$$

if κ is trivial. The summation is over characters θ of $F^\times$ for which $\theta^2=1$.

The integrals $F_1(\kappa,2,f_Y)$ and $F_1(1,2,f_Y)$ of (4.8) and the two integrals contributing to $F_1(1,2,f_Y)$ in (4.9) (cf. (2.10)) are each of the form (1.20) and hence vanish.

Since $\zeta=s^2(t_0^2A_1B)$ and $t_0^2A_1B=t_0\sigma(t_0)B$ the formula (2.10) yields

$$F_1(\theta,1,f_Y)=\oint_{E_0(F)}\frac{\kappa(B)}{\theta(B)}\frac{\kappa(B-1)}{|B-1|^2}\frac{1}{\hat\theta(A_1)}\frac{dA_1}{|A_1|}\,dB, \tag{4.10}$$

where $\hat\theta(A_1)=\theta(t_0\sigma(t_0))$. Recall that A_1 ranges over $\{x\in L^\times\colon \mathrm{Nm}_F^L x=1\}$. The character $\hat\theta$ is trivial if and only if $\theta\equiv\kappa$ or $\theta\equiv\kappa\theta_L$. In the case $\hat\theta\not\equiv 1$,

$$\int\frac{dA_1}{\hat\theta(A_1)|A_1|}=0\quad\text{(cf. Remark 1.G)}$$

and so (4.10) vanishes. In the case $\theta\equiv\kappa$,

$$\oint_{\mathbf{P}^1(F)}\frac{\kappa(B-1)}{|B-1|^2}\,dB=\oint_{\mathbf{P}^1(F)}\frac{\kappa(B)}{|B|^2}\,dB=0\quad\text{(Lemma 1.C)}$$

and again (4.10) is zero. Thus only $\theta\equiv\kappa\theta_L$ may give a nonzero contribution. The expansions (4.8) and (4.9) therefore take the form (4.6), and Lemma 4.B is proved.

Lemma 4.B and Lemma 4.A in the case G split over F imply Lemma 4.A for G anisotropic over F.

References

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