On unitary representations of the Virasoro algebra

The Virasoro algebra $b$ is an infinite-dimensional Lie algebra with basis $L_m, m \in \mathbb{Z}$, and $Z$ and defining relations:

(i) $[L_m, L_n] = (m - n)L_{m+n} + \frac{m(m^2-1)}{12}\delta_{m,-n}Z$

(ii) $[L_m, Z] = 0$

Some representations $\pi$ of $b$ of particular interest [2] are the Verma modules $(V, \pi) = (V^{h,c}, \pi^{h,c}), h, c \in \mathbb{R}$. They are characterized by the following conditions.

(i) There is a vector $v = v_\phi \neq 0$ in $V$ such that $L_nv = 0, n > 0, L_0v = hv, Zv = cv$.

(ii) Let $A$ be the set of sequences of integers $k_1, \geq k_2, \ldots \geq k_r > 0$ of arbitrary length, and if $\alpha \in A$ let $v_\alpha = \pi(L^{-k_1})\ldots\pi(L^{-k_r})v_\phi$. Then $\{v_\alpha | \alpha \in A\}$ is a basis of $V$.

Observe that $V$ is just the free vector space with basis $\{v_\alpha\}$ and is thus independent of $h$ and $c$. It is easy to see [1] that there is a unique sesquilinear form $\langle u, v \rangle = \langle u, v \rangle^{h,c}$ on $V$ with the properties:

(i) $\langle v_\phi, v_\phi \rangle = 1$;

(ii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$;

(iii) $\langle \pi(L_m)u, v \rangle = \langle u, \pi(L_{-m})v \rangle, m \in \mathbb{Z}$.

If this form is non-negative then the representation $\rho$ of $b$ on the quotient of $V$ by the space of null vectors is unitary, in the sense that

$$\rho(L_m)^* = \rho(L_{-m}).$$

**Theorem FQS.** The form $\langle \cdot, \cdot \rangle_{h,c}$ is non-negative only if either $c \geq 1, h \geq 0$ or there exists an integer $m \geq 2$ and two integers $p, q, 1 \leq p < m, 1 \leq q \leq p$, such that

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{(m+1)p - mq)^2 - 1}{4m(m+1)}.$$

This theorem has been proven by Friedan-Qiu-Shenker [1]. The sketch of the proof that they provided was unconscionably brief, and has evoked some scepticism among mathematicians. In this

note, which grew out of a series of lectures at the Centre de recherches mathématiques that overlapped the workshop, details are worked out. In the meantime, Friedan, Qiu and Shenker have themselves provided them [3], but the present account, which turns out to diverge from theirs in some respects, may still be a useful supplement to it. Several other authors have proven that the conditions of the theorem are not only necessary but also sufficient for non-negativity, but that is not the concern here.

The proof proceeds by lemmas. I write $Lv$ rather than $\pi(L)v, L \in \mathfrak{v}, v \in \mathfrak{v}$.

**Lemma 1.** If $(\cdot, \cdot)$ is non-negative then $h \geq 0, c \geq 0$.

**Proof.** Since $L_n L_{-n} v_\phi = L_{-n} L_n v_\phi + 2n h v_\phi + \frac{n(n^2-1)}{12} c v_\phi$, we have $\langle L_{-n} v_\phi, L_{-n} v_\phi \rangle = 2n h + \frac{n(n^2-1)}{12} c$.

Taking $n$ first equal to 1 and then very large we obtain the lemma.

For arbitrary $m$ we set $c = c(m) = 1 - \frac{6}{m(m+1)}, h_{p,q} = h_{p,q}(m) = \frac{(m+1)p-mq)^2-1}{4m(m+1)}, p, q \in \mathbb{N}$.

Observe that $c(-1-m) = c(m)$ and that $h_{p,q}(-1-m) = h_{q,p}(m)$.

**Lemma 2.**

(a) For $1 < c < 25, m$ is not real and neither is $h_{p,q}(m)$ unless $p = q$.

(b) As $m$ runs from 2 to $\infty$, $c$ increases monotonically from 0 to 1.

(c) For $c > 1, -1 < m < 0$.

(d) If $-1 < m < 0$ then $h_{p,q}(m) < 0$ unless $p = q = 1$ when $h_{p,q}(m) = 0$.

(e) If $p = q$ then $h_{p,q}(m) = \frac{p^2-1}{24} (1-c)$.

(f) If $p \neq q$ then $h_{p,q} + h_{q,p} = \frac{p^2+q^2-2}{24} (1-c) + \frac{(p-q)^2}{2}$.

In addition $h_{p,q} h_{q,p}$ is equal to

$$\frac{(p^2q^2 - p^2 - q^2 + 1)}{16 \cdot 36} (1-c)^2 + \frac{2p^2q^2 - pq(p^2 + q^2) - (p - q)^2}{48} (1-c) + \frac{(p^4 + q^4 - 4pq^3 - 4p^3q + 6p^2q^2)}{16}.$$
hermitian linear transformation $H_n = H_n(h, c) : \langle u, v \rangle_n = \{H_n u, v \}_n$. Let $P(n)$ be the dimension of $V_n$. It is the number of partitions of $n$. The Kac determinant formula (cf. [1]) is the key to the proof of Theorem FQS.

**Kac determinant formula.** If $c = c(m)$ then

$$\det H_n(h, c) = A_n \prod_{k \leq n} (h - h_{p,q})^{P(n-k)},$$

where $A_n$ is a positive constant.

**Lemma 3.** The form $\langle \cdot, \cdot \rangle_n$ is non-negative for $h \geq 0, c \geq 1$.

**Proof.** By continuity it suffices to treat pairs for which $h > 0, c > 1$. Since the previous lemma implies that $\det H_n(h, c)$ is nowhere zero in this region, it suffices to prove that the form is positive for one pair $(h, c)$. If $\alpha = (k_1, \ldots, k_r), r = r(\alpha), n(\alpha) = k_1 + \ldots + k_r$, set $v'_\alpha = L_{-k_r} \ldots L_{-k_1} v_\phi$. It is generally different than $v_\alpha$. It clearly suffices to show that for a given $c$ and $h$ large,

$$\langle v'_\alpha, v'_\alpha \rangle = c_\alpha h^{r(\alpha)} (1 + o(1)), \quad c_\alpha > 0 \quad (3.1)$$

$$\langle v'_\alpha, v'_\beta \rangle = o(h^{r(\alpha) + r(\beta)})/2), \quad \alpha \neq \beta. \quad (3.2)$$

This is proved by induction on $n(\alpha) + n(\beta)$. First of all $L_k^a L_{-k}^a$ is equal to

$$L_k^{a-1}(bL_0 + d)L_{-k}^{a-1} + L_k^{a-1} L_{-k} L_k L_{-k}^{a-1}, \quad b > 0.$$}

Moving the single $L_k$ in the second term ever further to the right, we obtain finally $L_k^a L_{-k}^a = L_k^{a-1}(bL_0 + d)L_{-k}^{a-1} + L_k^{a-1} L_{-k} L_k L_{-k}^{a-1}, \quad b > 0$. Take $k_1 \geq k_2 \geq \ldots \geq k_r > k$. If $\alpha = (k_1, \ldots, k_r, k, \ldots, k)$, then

$$\langle v'_\alpha, v'_\alpha \rangle = \langle L_{k_1} \ldots L_{k_r} L_k^a L_{-k_r} \ldots L_{-k_1} v_\phi, v_\phi \rangle$$

$$= c_{k_1, \alpha} h(1 + o(h))\langle L_{k_1} \ldots L_{k_r} L_k^{a-1} L_{-k_r} \ldots L_{-k_1} v_\phi, v_\phi \rangle$$

$$+ \langle L_{k_1} \ldots L_{k_r} L_k^{a-1} L_{-k_r} L_k L_{-k_r} \ldots L_{-k_1} v_\phi, v_\phi \rangle$$

with $c_{k_1, \alpha} > 0$. In the second term we move the $L_k$ further and further to the right obtaining the sum of
\( (k + k_r) \langle L_{k_1} \ldots L_{k_r} L_k^{-a-1} L_{-k}^a L_{-k-1} \ldots L_{-k_{j+1}} L_{-(k_{j-1})} \ldots L_{-k_1} v_\phi, v_\phi \rangle. \)

The induction assumption together with the defining relations for \( \nu \) implies readily that each of these terms is \( o(h^{r(\alpha)}) \) and that

\[
\langle L_{k_1} \ldots L_{k_r} L_k^{-a-1} L_{-k}^a L_{-k-1} \ldots L_{-k_{j+1}} L_{-(k_{j-1})} \ldots L_{-k_1} v_\phi, v_\phi \rangle = \langle \nu_\gamma, \nu_\gamma' \rangle = c_\gamma h^{r(\gamma)} (1 + o(1)),
\]

if \( \gamma = (k_1, \ldots, k_r, k, \ldots, k) \), with \( k \) repeated \( a - 1 \) times, so that \( r(\alpha) = 1 + r(\gamma) \).

On the other hand, if \( \beta = (\ell l_1, \ldots, \ell l_s, k, \ldots, k) \), with \( k \) repeated \( a' \leq a \) times, \( a > 0, a' \geq 0, \ell l_s \geq k \) even if \( a' = 0 \), then

\[
\langle \nu_\beta, \nu_\alpha' \rangle = \langle L_{k_1} \ldots L_{k_r} L_k^{a'} L_{-\ell l_1} \ldots L_{-\ell l_1} v_\phi, v_\phi \rangle
\]

is equal to the sum of

\[
c_{k,a'} h(1 + o(1)) \langle L_{k_1} \ldots L_{k_r} L_k^{-a-1} L_{-k}^a L_{-\ell l_1} \ldots L_{-\ell l_1} v_\phi, v_\phi \rangle
\]

and

\[
\Sigma_j (k + \ell l) \langle L_{k_1} \ldots L_{k_r} L_k^{-a'-1} L_{-k}^a L_{-\ell l_1} \ldots L_{-\ell l_{j+1}} L_{-(\ell l_{j} - k)} \ldots L_{-\ell l_{j-1}} \ldots L_{-\ell l_1} v_\phi, v_\phi \rangle.
\]

We take \( c_{k,0} = 0 \) if \( a' = 0 \). So induction yields (3.2).

Observe that if \( m > 0 \) and \( p > q \) then \( h_{p,q} > h_{q,p} \). If \( h \geq 0 \) and \( m > 0 \) define \( M > 0 \) by

\[
M^2 = 1 + 4m(m+1)h.
\]

Then \( M \geq 1 \). Let \( D \) be the closed shaded region in the diagram I. It is bounded by the lines

\[
mx - (m + 1)y = \pm M \text{ and } (m + 1)x - my = M.
\]
Lemma 4.

(a) \( h_{p,q} \geq h \geq h_{q,p} \) if and only if \((p,q) \in D\).

(b) \( D \) contains an integral point \((p,q)\) with \(q > 0\).

**Proof.** Since \( h_{p,q} \geq h \) if and only if \(((m+1)p - mq)^2 \geq M^2 \) and \(h \geq h_{q,p}\) if and only if \(((m+1)q - mp)^2 \leq M^2\), the first statement of the lemma is clear. For the second choose a large integer \( p \) and let \( a = \frac{mp-M}{m+1} \). Then the points \((p,q)\) with \( a \leq q \leq a + \frac{2M}{m+1} \) lie in \( D \). So do the points \((p+1,q), a + \frac{m}{m+1} \leq q \leq a + \frac{m+2M}{m+1} \) and so on. So we need only show that one of the intervals \([a + \frac{km}{m+1}, a + \frac{km+2M}{m+1}], k \in \mathbb{Z}, k \geq 0\), contains an integer. This is clear if \( \frac{m}{m+1} \) is irrational. Otherwise, increasing \( q \) if necessary, we may suppose that \( a \) is as close to its integral part as any \( a + \frac{km}{m+1} \). Then \( a + \frac{m}{m+1} < [a] + 1 \), but \( a + \frac{m+2M}{m+1} \geq a + \frac{m+2}{m+1} > [a] + 1 \), and the interval \([a + \frac{m}{m+1}, a + \frac{m+2M}{m+1}]\) contains \([a] + 1\).

Let \( p(h,c) = \min_{(p,q) \in DP} p \) and let \( q(h,c) = \min_{(p,q) \in Dq} q \). It is clear that

\[
P(h,c) = (p(h,c), q(h,c)) \in D.
\]

In the following geometrical arguments, it is sometimes necessary to recall that \( h - h_{p_0,p_0} < 0 \) if and only if \( p_0 > M \).
Lemma 5. If $P(h,c)$ lies in the interior of $D$ then $(v,v)$ assumes negative values in $V$.

Proof. Let $(p,q) = P(h,c)$ and let $n = pq$. If $p_0 q_0 \leq n$, $p_0 \geq q_0$ and $(p_0, q_0) \neq (p, q)$ then either $p_0 < p$ or $q_0 < q$ so that $(p_0, q_0) \notin D$. In general set

$$
\phi_{p_0, q_0} = (h - h_{p_0, q_0}), \quad p_0 \neq q_0,
$$

$$
= h - h_{p_0, q_0}, \quad p_0 \neq q_0.
$$

If $(p_0, q_0) \notin D$ and $p_0 \neq q_0$ then $\phi_{p_0, q_0} > 0$.

Suppose that for some $p_0$ with $p_0^2 \leq pq$ we had $h - h_{p_0, p_0} < 0$. Then there would be a minimum such $p_0$ and if $n_0 = p_0^2$ then

$$
\det H_{\alpha_0} = A_{\alpha_0} \prod_{\substack{p_1 \geq q_1, \\
n_1 = p_1 q_1 \leq n_0}} \phi_{p_1, q_1}^{P(n_0-n_1)}
$$

Since $P(h,c)$ lies in the interior of $D$, $p \neq q$ and none of the pairs $(p_1, q_1)$ that intervene here lie in $D$. Moreover, all terms of the products are positive save $\phi_{p_0, p_0}^{P(0)} = \phi_{p_0, p_0}$. Since this is negative, $\langle \cdot, \cdot \rangle$ assumes negative values on $V_{n_0}$.

If, however, $\phi_{p_0, p_0} > 0$ for all $p_0 \leq q$ then the same argument shows that $\det H_n < 0$, so that $\langle \cdot, \cdot \rangle$ assumes negative values on $V_n$.

The treatment of those points $(h, c)$ for which $P(h, c)$ lies on the boundary of $D$ is more delicate. There are at first three possibilities for $(p, q) = P(h, c)$:

(A) $mp - (m + 1)q = M$;

(B) $(m + 1)p - mq = M$;

(C) $mp - (m + 1)q = -M, p \neq q$;

Lemma 6. The case (C) above does not occur.

Proof. It is clear from the diagram defining $D$ that in case (C), $p \geq M, q \geq M$. If $q = 1$ then $M = 1$ and $p = 1$, so that we have rather case (B). If $q > 1$ then $p > 1$ and $(m + 1)(q - 1) - m(p - 1) = (m + 1)q - mp - 1$, so that $M > (m + 1)(q - 1) - m(p - 1) > -M$. Moreover, $(m + 1)(p - 1) - m(q - 1) - M = (m + 1)(p - 1 - q) - m(q - 1 - p) = (2M + 1)(p - q) - 1$. Since $m \geq 2$ this is positive if $p \neq q$. Consequently $(p - 1, q - 1) \in D$, and this is a contradiction.

Fix $(p, q)$. In case (A) we have $h = h_{q,p}(m), c = c(m)$. In case (B) we have $h = h_{p,q}(m), c = c(m)$.
Lemma 7.
(a) The set of all \( m \geq 2 \) for which \( h = h_{q,p}(m), c = c(m) \) yields case (A) is the interval \( m > q + p - 1 \).

(b) The set of all \( m \geq 2 \) for which \( h = h_{p,q}(m), c = c(m) \) yields case (B) is the interval \( m > q + p - 1 \) if \( (p, q) \neq (1, 1) \) and is the interval \( m \geq 2 \) if \( (p, q) = (1, 1) \).

It will be helpful, when proving this and the following lemmas, to keep the diagrams IIA and IIB in mind.

Diagram IIA

**Proof.** We first show that if \( h_{q,p}(m_0), c(m_0) \) yield case (A) then so does \( (h_{q,p}(m), c(m)) \) for \( m \geq m_0 \). It is clear from the diagram that it is sufficient to verify that \( M, \frac{M}{m+1}, \) and \( \frac{M}{m} \) are increasing functions of \( m \). But \( M = m(p - q) - q, \frac{M}{m} = (p - q) - \frac{q}{m}, \frac{M}{m+1} = (p - q) - \frac{p}{m+1} \). It is also clear that
we can decrease \( m \) without passing out of case (A) so long as \( M = m(p - q) - q \) remains greater than or equal to 1 and \((m + 1)(p - 1) - mq > mp - (m + 1)q\). But

\[
(m + 1)(p - 1) - mq = mp - (m + 1)q \iff m = p + q - 1.
\]

As we decrease to these points, \( M \) decreases to

\[
(p + q - 1)(p - q) - q = p^2 - q^2 - p = (p - 1)^2 - q^2 + p - 1.
\]

This number is greater than 1 because \( p > q \geq 1 \).

For case (B), \( M = m(p - q) + p \) is a non-decreasing function of \( m \), and \( \frac{M}{m} = (p - q) + \frac{p}{m} \cdot \frac{M}{m+1} = (p - q) + \frac{q}{m+1} \) are decreasing functions. Since the slope of \( mp - (m + 1)q = M \) is \( 1 - \frac{1}{m+1} \), it is increasing and the conclusion is the same. The minimal value of \( m \) is given by

\[
(m + 1)p - mq = mp - (m + 1)(q - 1) \iff m = p + q - 1.
\]

because
\[(p + q - 1)(p - q) + p = p^2 - q^2 + q \geq 1,\]

unless \(p = q = 1\) when \(m\) cannot go below 2.

In case (A) the intersection of the two lines \((m + 1)x - my = M\) and \(x - y = p - q - 1\) is a point \((x(m), y(m))\) with \(p' \geq x(m) > p' - 1\) where \(p'\) is an integer, \(p' \geq p\). If \(x(m) = p'\) then \(y(m) = q' = p' - p + q + 1,\) and \(m = p' + q.\) Thus \(m \in \{2, 3, \ldots\}, p' < m, q' \leq p'\) and \(c = c(m), h = h_{p', q'}(m).\)

In case (B) the intersection of the lines \(x - y = p - q + 1\) and \(mx - (m + 1)y = M\) is a point \((x(m), y(m))\) with \(p' \geq x(m) > p' - 1, p' - 1 \geq p.\) If \(x(m) = p'\) then \(m = p + q'\) lies in \(\{2, 3, \ldots\}, q \leq p, p < m\) and \(c = c(m), h = h_{p, q}(m).\)

Thus to prove the theorem it suffices to establish the following proposition.

**Proposition.** If case (A) or (B) obtains and \(p' > x(m) > p' - 1\) then the form \(\langle \cdot, \cdot \rangle\) assumes negative values in \(V.\)

We assume the contrary and derive a contradiction. We occasionally abbreviate \(c(m)\) to \(c\) and \(h_{q, p}(m)\) or \(h_{p, q}(m)\) to \(h(m)\) or to \(h.\)

**Lemma 8.**

(a) Suppose \(p' > x(m) > p' - 1.\) If \((p_1, q_1)\) lies on the boundary of \(D(h, c)\) and \(p_1q_1 \leq p'q'\) then \((p_1, q_1) = (p, q).\)

(b) Define \(m'\) by \(p' = x(m')\) and set \(c' = c(m'), h' = h_{q, p}(m')\) or \(h_{p, q}(m').\) If \((p_1, q_1)\) lies on the boundary of \(D(h', c')\) and \(p_1q_1 \leq p'q'\) then \((p_1, q_1)\) is \((p, q)\) or \((p', q').\)

**Proof.** Set \((x, y) = (x(m), y(m))\) and define \(z, z'\) as indicated by the diagrams. It clearly suffices to show that in case (A) \(y - z < 2, z' - x < 2,\) and that in case (B), \(x - z < 2, z' - y < 2.\) In case (A) elementary algebra yields \(m = x + q, y - z = \frac{x + z}{m} = 1 + \frac{y - q}{x + q}\) and \(\frac{y - q}{x + q} = \frac{z - y}{x - p} \cdot \frac{z - q}{x - p} < 1.\) On the other hand \(z' - x = \frac{x + y}{m} = 1 + \frac{y - q}{p + y - 1} < 2.\) A similar argument works for case (B).

Since \(p, q\) and \(p'\) are fixed it will be useful to let \(C\) denote the curve \(c = c(m), h = h_{q, p}(m)\) (A) or \(h = h_{p, q}(m)\) (B), \(m > p' - 1.\)
Lemma 9.

(a) If \( x(m) > p' - 1, x(m) \neq p' \), and \( n_1 \leq n' \), then the dimension of the space of null vectors in \( V_{n_1} \) is \( P(n_1 - n) \).

(b) If \( x(m) = p' \) and \( n_1 < n' \) then the dimension of the space of null vectors in \( V_{n_1} \) is \( P(n_1 - n) \), but if \( n_1 = n' \) it is \( P(n_1 - n) + 1 \).

**Proof.** Observe that \( P(n_1 - n) = 0 \) if \( n_1 < n \) and that when this is so the lemma is clear. So take \( n_1 \geq n \) and denote the pertinent dimension by \( d_{n_1}^0 \). We begin by showing that \( d_{n_1}^0 > 0 \) and that \( d_{n_1}^0 \leq P(n_1 - n) \) unless \( x(m) = p' \) and \( n_1 = n' \) when \( d_{n_1}^0 \leq P(n_1 - n) + 1 \).

For \( 0 \leq c < 1, m \) is locally an analytic function of \( c \) and we may write \( h_{p,q}(m) = h_{p,q}(c) = h(c) \) or \( h_{q,p}(m) = h_{q,p}(c) = h(c) \). Fix \( c \) and consider \( H_{n_1}(h, c) \) as a function of \( h \) near \( h(c) \). Its eigenvalues are the roots of a polynomial equation with real analytic, indeed polynomial, coefficients and they are all real for \( h \) real. It is easily seen that this implies that there is no ramification at \( h = h(c) \) and that in a neighborhood of this point there are expansions

\[
\alpha_i(h) = \alpha_{i0} + \alpha_{i1}(h - h(c)) + \alpha_{i2}(h - h(c))^2 + \ldots, \quad 1 \leq i \leq P(n_1)
\]

for the eigenvalues of \( H_{n_1} \). Thus

\[
\det H_{n_1}(h, c) = \Pi_{i=1}^{P(n_1)} (\alpha_{i0} + \alpha_{i1}(h - h(c)) + \ldots),
\]

and the power of \( h - h(c) \) that divides it is greater than or equal to the number of zero eigenvalues of \( H_{n_1}(h(c), c) \). On the other hand, the left side is equal to

\[
A_n \Pi_{k \leq n_1} \Pi_{p_1, q_1 = k} (h - h_{p_1, q_1}(c))^P(n_1 - k),
\]

and \( h_{p_1, q_1}(c) = h(c) \) only if \( (p_1, q_1) \) or \( (q_1, p_1) \) lies in boundary of \( D \). Thus the assertion follows from Lemma 8.

Choosing \( n_1 = n \), we see in particular that the dimension of the null space of \( V_n \) is 1. Thus if \( m > p' - 1 \) then in a neighborhood of \( (h(m), c(m)) \) we can find an analytic function \( v(h, c) \) with values in \( V_n \) such that \( v(h, c) \) has length 1, is an eigenvector of \( H_n(h, c) \), and corresponds to the eigenvalue 0 when \( (h, c) \) falls on the curve \( C \).
Since

\[ L_0 v(h(m), c(m)) = (h(m) + n)v(h(m), c(m)), \]

\[ L_k v(h(m), c(m)) = 0, \quad k > 0, \]

there is a homomorphism of \( v \)-modules, \( \phi : V^{h(m)+n,c(m)} \to V^{h(m),c(m)} \), taking \( v_{\phi}^{h(m)+n,c(m)} \) to \( v(h(m), c(m)) \). If it is injective on \( V^{h(m)+n,c(m)} \) then \( d_{n_1}^0 \geq P(n_1 - n) \) because the image consists of null vectors. Since \( d_{n_1}^0 \) is lower semicontinuous, \( d_{n_1}^0 \) will be greater than or equal to \( P(n_1 - n) \) everywhere on \( C \) if it is so on a dense set. The homomorphism \( \phi \) will be injective if \( \det H_{n_1-n}^{h(m)+n,c(m)} \neq 0 \) because the kernel consists of null vectors. So it is enough to show that this determinant does not vanish identically on \( C \). However, if \( h(m) + n = h_{p_1,q_1}(m) \) then

\[ ((m+1)p + mq)^2 = ((m+1)p_1 - mq_1)^2 \]

or

\[ (mp + (m+1)q)^2 = ((m+1)p_1 - mq_1)^2. \]

This can occur for at most two values of \( m \).

It remains to show that at \( m' \) the dimension of the space of null vectors in \( V_{n'} \) is \( P(n' - n) + 1 \). For this we need further lemmas.

**Lemma 10.** \( \det H_{n'-n}^{h(m')+n,c(m')} \neq 0. \)

**Proof.** It has to be shown that equality \( h(m') + n = h_{p_1,q_1}(m'), p_1q_1 \leq n' - n \) is impossible. This equality amounts to

\[ (m'p + (m'+1)q)^2 = ((m'+1)p_1 - m'q_1)^2 \quad (A) \]

or

\[ ((m'+1)p + m'q)^2 = ((m'+1)p_1 - m'q_1)^2. \quad (B) \]

It is not supposed that \( p_1 \geq q_1. \)
The first equation implies that \( m'p + (m' + 1)q = \pm((m' + 1)p_1 - m'q_1) \) or \( m'(p \pm q_1) = (m' + 1)(\pm p_1 - q) \). Since \( m' \) is an integer this implies \( (p \pm q_1) = a(m' + 1), (\pm p_1 - q) = am' \). Since \( n' = p'q' = (m' - q)(m' - p + 1) \) the inequality \( n' \geq n + p_1q_1 \) becomes

\[
(m' - q)(m' - p + 1) \geq a(m' + 1)q - am'p + a^2m'(m' + 1)
\]
or

\[
((1 + a)(m' + 1) - p)((1 - a)m' - q) \geq 0.
\]

Since \( m' = p' + q = p + q' - 1, m' > q, m' + 1 > p \). So the inequality is possible only for \( a = 0 \), but \( a \) cannot be 0. The case \( B \) is treated in a similar fashion.

For \( n_1 < n' \) or \( m \neq m' \) we let \( U_{n_1} = U_{n_1}(m) \) be the space of null vectors in \( V_{n_1} \). For \( h, c \) close to \( h(m'), c(m') \) we let \( U_{n_1}(h, c) \) be the span of

\[
\{ L_{-k_1} \ldots L_{-k_r}v(h, c)| k_1 \geq \ldots \geq k_r > 0, \Sigma k_i = n' - n \}.
\]

We set \( U_{n'}(m) = U_{n'}(h(m), c(m)) \), the two definitions of \( U_{n'}(m) \) coinciding when they both apply. Thus for \( m > p' - 1, U_{n_1}(m) \) is defined and analytic as a function of \( m \). Let \( W_{n_1} \) be its orthogonal complement with respect to the form \( \langle \cdot, \cdot \rangle \). It follows from that part of Lemma 9 already proved that the restriction \( J_{n_1} = J_{n_1}(m) \) of \( H_{n_1} \) to \( W_{n_1} \) is non-singular unless \( n_1 = n', m = m' \). In particular, our assumption, which was made for a particular \( m \), implies that \( J_{n_1}(m) \) is positive for all \( m > p' - 1 \) if \( n_1 < n' \).

**Lemma 11.** Near \( m' \), \( \det J_{n'}(m) = \delta(m)(m - m') \) where \( \frac{1}{\delta} \geq |\delta(m)| \geq \delta > 0 \).

It will follow from this lemma that the remaining assertion of Lemma 9 is true. In addition the lemma together with our assumption on the non-negativity of \( \langle \cdot, \cdot \rangle \) for a particular \( m, p' > x(m) > p' - 1 \), will imply that the form takes negative values for \( m > m' \) because \( \det J_{n'}(m) \) changes sign at \( m' \).

Let \( v(h, c) \), defined in a neighborhood of \( (h(m'), c(m')) \), correspond to the eigenvalue \( \alpha(h, c) \) of \( H_n(h, c) \). All the other eigenvalues of \( H_n(h, c) \) are bounded above and, if the neighborhood is sufficiently small, away from 0. On the other hand, all factors \( h - h_{p_1,q_1}(c) = h - h_{p_1,q_1}(m), c = c(m) \), of \( \det H_n(h, c) \) are bounded away from 0 in a neighborhood of \( h(m'), c(m') \) except for \( h - h(c) \), where
To verify this it is sufficient, for the pertinent basis can be supposed to depend analytically on $h, c$ to verify that they are zero when $h$ is a transformation $O$.

I claim that the entries in the off-diagonal blocks are bounded in absolute value above and below. So it is enough to consider $U(h,c)$ orthogonal with respect to the form $\langle \cdot, \cdot \rangle_n$. However the basis $\{\phi(v_\alpha)|v_\alpha \in V^{h(m)+n,c(m)}, n(\alpha) = n' - n\}$ is related to such a basis by a matrix whose determinant is bounded in absolute value above and below. So it is enough to consider $\det(\{\phi(v_\alpha), \phi(v_\beta)\})$.

We have

$$
\langle \phi(v_\alpha), \phi(v_\beta) \rangle = \langle L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), v(h,c) \rangle \\
= \{L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), H_{n'}(h,c) v(h,c) \rangle \\
= \alpha(h,c) \{L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), v(h,c) \rangle.
$$

At $h(m), c(m)$ the value of $\det(\{L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), v(h,c) \rangle)$ is

$$
\det((v_\alpha, v_\beta)^{h(m)+n,c(m')})_{n' - n}.
$$

By Lemma 10 this is not 0. Lemma 13 follows.

In a neighborhood of $h(m), c(m)$ we decompose $V_{n'}$ as an orthogonal sum $U_{n'} \oplus W_{n'}$. The linear transformation $H_{n'}(h,c)$, or its matrix with respect to a compatible basis, then decomposes into blocks.

I claim that the entries in the off-diagonal blocks are $O(h - h_{p,q}(c))$ in a neighborhood of $h(m), c(m)$. To verify this it is sufficient, for the pertinent basis can be supposed to depend analytically on $h, c$, to verify that they are zero when $h = h_{p,q}(c)$, but that is clear by the definition of $U_{n'}$.

It follows that

**Lemma 12.** In a neighborhood of $(h(m'), c(m'))$ we have $\alpha(h,c) = a(h,c)(h - h(c))$ with $\frac{1}{a} \geq |a(h,c)| \geq a > 0$, $a$ being a constant.

Here $h(c)$ is $h_{q,p}(m)$ ($A$) or $h_{p,q}(m)$ ($B$), $c = c(m)$. More generally we have

**Lemma 13.** Let $K_{n'}(h,c)$ be the restriction of $H_{n'}(h,c)$ to $U_{n'}(h,c)$. Then, in a neighborhood of $(h(m'), c(m'))$, $\det K_{n'}(h,c) = k(h,c)\alpha(h,c)^{P(n'-n)}$, with $\frac{1}{k} \geq |k(h,c)| \geq k > 0$.

**Proof.** The determinant of $K_{n'}(h,c)$ is that of the form $\langle \cdot, \cdot \rangle_{n'}$, calculated with respect to a basis of $U_{n'}(h,c)$ orthogonal with respect to the form $\{\cdot, \cdot\}_n$. However the basis $\{\phi(v_\alpha)|v_\alpha \in V^{h(m)+n,c(m)}, n(\alpha) = n' - n\}$ is related to such a basis by a matrix whose determinant is bounded in absolute value above and below. So it is enough to consider $\det(\{\phi(v_\alpha), \phi(v_\beta)\})$.

We have

$$
\langle \phi(v_\alpha), \phi(v_\beta) \rangle = \langle L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), v(h,c) \rangle \\
= \{L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), H_{n'}(h,c) v(h,c) \rangle \\
= \alpha(h,c) \{L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), v(h,c) \rangle.
$$

At $h(m), c(m)$ the value of $\det(\{L_{ell_1} \ldots L_{ell_k} L_{-k_1} \ldots L_{-k_r}, v(h,c), v(h,c) \rangle)$ is

$$
\det((v_\alpha, v_\beta)^{h(m)+n,c(m')})_{n' - n}.
$$

By Lemma 10 this is not 0. Lemma 13 follows.
\[ \det H_{n'}(h, c) = \det J_{n'}(h, c) \det K_{n'}(h, c) + O((h - h_{p,q}(c))^{P(n'-n)+1}) \]  

(1)

If \( J_{n'}(h, c) \) is the matrix in the diagonal block corresponding to \( W_{n'} \). Since

\[ \det H_{n'}(h, c) = A_{n'} \Pi_{k \leq n'} \Pi_{p,q_1 = k}(h - h_{p,q_1}(c))^{P(n'-p_1q_1)} \]

we may divide the relation (1) by \( (h - h_{p,q_1}(c))^{P(n'-n)} \) and then set \( h = h_{p,q}(c), c = c(m) \). The result clearly yields Lemma 11 because \( h(m') = h_{p_1,q_1}(m'), p_1, q_1 \leq n', \) only if \( (p_1, q_1) \) is \( (q, p) \) or \( (p', q') \) (case A) or \( (p, q) \) or \( (q', p') \) (case B).

Our assumption that \( H_{n_1}(h(m), c(m)) \) is non-negative for a given \( m, p' > m > p' - 1 \), has led to the conclusion that \( J_{n_1}(m) \) is positive for large \( m \) and \( n_1 < n' \) but that \( J_{n'}(m) \) has negative eigenvalues for large \( m \). We show not that this is impossible.

As \( m \) approaches infinity, the point \( (h(m), c(m)) \) approaches \( (h_0, c_0) = \left( \frac{(p-q)^2}{4}, 1 \right) \). If \( p \neq q \) a suitable coordinate on the curve is \( \mu = \frac{1}{m} \). If \( p = q \) we may take \( \mu = 1 - c \). All the matrices \( H_{n_1}(\mu) = H_{n_1}(m) = H_{n_1}(h(m), c(m)) \) are analytic functions of \( \mu \). The eigenvalues of \( H_{n_1}(\mu) \) are given by power series.

\[ \alpha_i = \alpha_i(\mu) = \alpha_{i0} + \alpha_{i1} \mu + \alpha_{i2} \mu^2 + \ldots \]

Let \( V^1_{n_1}(\mu) \) be the space spanned by the eigenvectors corresponding to \( \alpha_i \) with \( \alpha_{i0} = 0 \); let \( V^2_{n_1}(\mu) \) be the space spanned by the eigenvectors corresponding to \( \alpha_i \) with \( \alpha_{i0} = \alpha_{i1} = 0 \) and so on. One proves by induction that these spaces are well defined, depend analytically on \( \mu \) (in the sense that we have analytic functions \( v_1(\mu), \ldots, v_{p(n_1)}(\mu) \), such that \( \{v_1(\mu), \ldots, v_{d_1}(\mu)\} \), \( d_k \) = \( \dim V^k_{n_1}(\mu) \) forms a basis of \( V^k_{n_1}(\mu) \) for each \( \mu \), and that \( \mu^{-k} \{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\}, i \leq d_k, j \leq P(n_1) \) is analytic for small \( \mu \). It can even be supposed that \( \{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\} = 0, i \leq d_k, j > d_k \).

Let \( V^k = \oplus_{n_1} V^k_{n_1}(0) \) and \( X^k = V^k/V^{k+1} = \oplus_{n_1} V^k_{n_1}(0)/V^{k+1}_{n_1}(0) \). If \( u = \sum_{i \leq d_k} a_i v_i(0) \in V^k_{n_1}(0) \) and \( v = \sum_{i \leq d_k} b_i v_i(0) \in V^k_{n_2}(0) \), define \( (u, v)^{(k)} \) to be 0 if \( n_1 \neq n_2 \), and if \( n_1 = n_2 \) set

\[ (u, v)^{(k)} = (u, v)_{n_1}^{(k)} = \sum_{i} \tilde{a}_i \tilde{b}_j \lim_{\mu \to 0} \mu^{-k} \langle v_i(\mu), v_j(\mu) \rangle \]

\[ = \sum_{i} \tilde{a}_i \tilde{b}_j \lim_{\mu \to 0} \{\mu^{-k}H_{n_1}(\mu)v_i(\mu), v_j(\mu)\} \]

It is clear that \( H_{n_1}(\mu) \) is non-negative for small \( \mu \) if and only if the forms \( (u, v)^{(k)}_{n_1} \) are all positive.
Lemma 14.

(a) The spaces $V^k$ are all invariant under $\pi = \pi^{h_0, c_0}$, so that $v$ operates on $X^k$.
(b) The form $< \cdot, \cdot >^{(k)}$ on $X^k$ satisfies $< L_m x, y > = < x, L_{-m} y >$, $m \in \mathbb{Z}$.

Proof. Set $L_m(\mu) = \pi^{h(\mu), c(\mu)}(L_m)$ and $L_m = L_m(0)$. We have to show for each $n_1$ that $L_m v_i \in V^k$ if $v_i = v_i(0)$ and $i \leq d_k$. However

$$L_m v_i = \lim_{\mu \to 0} L_m(\mu) v_i(\mu) = \lim_{\mu \to 0} \Sigma_{j} c_{ij}(\mu) v_j'(\mu)$$

where the $c_{ij}$ are analytic functions of $\mu$. It is to be shown that $c_{ij}(0) = 0$ for $j > d_k$. The primes refer to $n_2 = n_1 - m$ rather than to $n_1$. In other words it has to be shown that $\{ H_{n_2}(\mu) L_m(\mu) v_i(\mu), v_j'(\mu) \} = O(\mu^k)$ for all $\ell$. Since $H_{n_2}(\mu) L_m(\mu) = L_{-m}(\mu) H_{n_1}(\mu)$, the adjoint of $L_{-m}(\mu)$ being taken with respect to the form $\{ \cdot, \cdot \}$, this is clear. So is the second assertion of the lemma.

For any $h \geq 0$ the representation $\pi^{h, 1}$ on $V^{h, 1}$ has a unique irreducible quotient $\rho^{h, 1}$ on $X^{h, 1}$, which by Lemma 3 carries a hermitian form for which $\rho^{h, 1}$ is unitary in the sense that the adjoint $\rho^{h, 1}(L_m)$ is $\rho^{h, 1}(L_{-m})$. Such a form is unique up to a scalar multiple. Take in particular $h = \frac{r^2}{4}, r \in \mathbb{Z}$. Then $h = h_{p_2, q_2}(c)$ if and only if $(p_2 - q_2)^2 = r^2$. In particular, $h = h_{r + 1, 1}(c)$. Thus the lowest weight for a null vector in $V$ is $r + 1$ and $h + r + 1 = \frac{(r + 2)^2}{4}$, so that the kernel of $V^{h, 1} \to X^{h, 1}$ contains a quotient of $V^{h', 1}, h' = \frac{(r + 2)^2}{4}$. Thus $V^{h, 1}$ admits a sequence of invariant subspaces $V^{h, 1} \supseteq V^{h, 1}(1) \supseteq V^{h, 1}(2)$ such that the representation on $V^{h, 1}(0)/V^{h, 1}(1)$ is $\rho^{h, 1}$ and that on $V^{h, 1}(1)/V^{h, 1}(2)$ is $\rho^{h', 1}$. In general set $h^{(\ell)} = \frac{1}{4}(r + 2\ell)^2$.

Lemma 15. $V^{h, 1}$ admits an infinite decomposition series $V^{h, 1}(0) \supseteq V^{h, 1}(1) \supseteq \ldots \supseteq V^{h, 1}(\ell) \supseteq \ldots$ such that the representation on the quotient $V^{h, 1}(\ell)/V^{h, 1}(\ell + 1)$ is $\rho^{h^{(\ell)}, 1}$.

Proof. If $\lambda = h + k, k \in \mathbb{Z}, k \geq 0$, let $d_{\lambda} = \dim\{ v \in V^{h, 1} | L_0 v = \lambda v \}$, $d_{\lambda}(\ell) = \dim\{ v \in X^{h^{(\ell)}, 1} | L_0 v = \lambda v \}$. The lemma follows easily from a formula of Kac([2], Th. 5), according to which $d_{\lambda} = \Sigma_{\ell=0}^{\infty} d_{\lambda}(\ell)$. Indeed, suppose we have constructed an initial segment of the series $V^{h, 1}(0) \supseteq \ldots \supseteq V^{h, 1}(\ell)$. Then $\frac{1}{4}(r + 2\ell)^2$ is a lowest weight in $V^{h, 1}(\ell)$ and $\dim\{ v \in V^{h, 1}(\ell) | L_0 v = \frac{1}{4}(r + 2\ell)^2 \} = 1$. Take $V^{h, 1}(\ell + 1)$ to be the sum of all invariant subspaces of $V^{h, 1}(\ell)$ for which the lowest weight is greater than $\frac{1}{4}(r + 2\ell)^2$. 
Now take \( r = p - q \). It follows immediately from the preceding lemma that \( X^k_j \) is the direct sum of irreducible invariant subspaces \( X^k_j \) carrying distinct representations and that the restriction of \( \langle \cdot, \cdot \rangle^k \) to \( X^k_j \) is either positive or negative. The assumption that we are trying to contradict implies that the form is positive if \( X^k_j \) contains non-zero vectors of weight \( h + n_1, n_1 < n' \), but that for some \( j \) and \( k \) for which \( X^k_j \) contains vectors of weight \( h + n' \), it is negative.

Thus the following lemma completes the proof of Theorem FQS.

**Lemma 16.** The equation \( \frac{r^2}{4} + n' = \frac{1}{4}(r + 2\ell')^2 \) has no solution \( \ell \geq 0 \) in \( \mathbb{Z} \).

**Proof.** The equation may be written as \( n' = \ell\ell + r \). Recall that \( n' \) is \( (p+a)(q+a+1) \) in case A and \( (p+a+1)(q+a) \) in case B, with \( a \geq 0 \). Since \( r = p-q \), the equation is \( (p+a+\ell)(q+a+1-\ell) = \ell \) or \( (p+a+1-\ell)(q+a-\ell) = -\ell \). Both equations are manifestly impossible.

**References**

