On unitary representations of the Virasoro algebra*

The Virasoro algebra $\mathfrak{b}$ is an infinite-dimensional Lie algebra with basis L_m, $m \,\epsilon\, \mathbf{Z}$, and Z and defining relations:

(i) $[L_m, L_n] = (m-n)L_{m+n} + \frac{m(m^2-1)}{12}\delta_{m,-n}Z$;

(ii) $[L_m, Z] = 0$.

Some representations π of $\mathfrak{b}$ of particular interest [2] are the Verma modules $(V, \pi) = (V^{h,c}, \pi^{h,c}), h, c \,\epsilon\, \mathbf{R}$. They are characterized by the following conditions.

(i) There is a vector $v = v_\phi \neq 0$ in V such that $L_n v = 0, n > 0$, $L_0 v = hv, Zv = cv$.

(ii) Let A be the set of sequences of integers $k_1, \geq k_2 \geq \ldots \geq k_r > 0$ of arbitrary length, and if $\alpha \,\epsilon\, A$ let $v_\alpha = \pi(L_{-k_1}) \ldots \pi(L_{-k_r})v_\phi$. Then $\{v_\alpha | \alpha \,\epsilon\, A\}$ is a basis of V.

Observe that V is just the free vector space with basis $\{v_\alpha\}$ and is thus independent of h and c. It is easy to see [1] that there is a unique sesquilinear form $\langle u, v\rangle = \langle u, v\rangle^{h,c}$ on V with the properties:

(i) $\langle v_\phi, v_\phi\rangle = 1$;

(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$;

(iii) $\langle \pi(L_m)u, v\rangle = \langle u, \pi(L_{-m})v\rangle, m \,\epsilon\, \mathbf{Z}$.

If this form is non-negative then the representation ρ of $\mathfrak{b}$ on the quotient of V by the space of null vectors is unitary, in the sense that

$$\rho(L_m)^* = \rho(L_{-m}).$$

Theorem FQS. *The form $\langle \cdot, \cdot\rangle_{h,c}$ is non-negative only if either $c \geq 1, h \geq 0$ or there exists an integer $m \geq 2$ and two integers $p, q, 1 \leq p < m, 1 \leq q \leq p$, such that*

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}.$$

This theorem has been proven by Friedan-Qiu-Shenker [1]. The sketch of the proof that they provided was unconscionably brief, and has evoked some scepticism among mathematicians. In this

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* Appeared in *Infinite-dimensional Lie algebras and their applications*, World Scientific (1988)

note, which grew out of a series of lectures at the Centre de recherches mathématiques that overlapped the workshop, details are worked out. In the meantime, Friedan, Qiu and Shenker have themselves provided them [3], but the present account, which turns out to diverge from theirs in some respects, may still be a useful supplement to it. Several other authors have proven that the conditions of the theorem are not only necessary but also sufficient for non-negativity, but that is not the concern here.

The proof proceeds by lemmas. I write Lv rather than $\pi(L)v, L \,\epsilon\, \mathfrak{v}, v \,\epsilon\, V$.

Lemma 1. *If $\langle\cdot,\cdot\rangle$ is non- negative then $h \geq 0, c \geq 0$.*

Proof. Since $L_n L_{-n} v_\phi = L_{-n} L_n v_\phi + 2nhv_\phi + \frac{n(n^2-1)}{12} cv_\phi$, we have $\langle L_{-n}v_\phi, L_{-n}v_\phi\rangle = 2nh + \frac{n(n^2-1)}{12}c$.

Taking n first equal to 1 and then very large we obtain the lemma.

For arbitrary m we set $c = c(m) = 1 - \frac{6}{m(m+1)}, h_{p,q} = h_{p,q}(m) = \frac{((m+1)p-mq)^2-1}{4m(m+1)}, p, q \,\epsilon\, \mathbf{N}$. Observe that $c(-1-m) = c(m)$ and that $h_{p,q}(-1-m) = h_{q,p}(m)$.

Lemma 2.

(a) *For $1 < c < 25, m$ is not real and neither is $h_{p,q}(m)$ unless $p = q$.*

(b) *As m runs from 2 to ∞, c increases monotonically from 0 to 1.*

(c) *For $c > 1, -1 < m < 0$.*

(d) *If $-1 < m < 0$ then $h_{p,q}(m) < 0$ unless $p = q = 1$ when $h_{p,q}(m) = 0$.*

(e) *If $p = q$ then $h_{p,q}(m) = \frac{p^2-1}{24}(1-c)$.*

(f) *If $p \neq q$ then $h_{p,q} + h_{q,p} = \frac{p^2+q^2-2}{24}(1-c) + \frac{(p-q)^2}{2}$.*

In addition $h_{p,q}h_{q,p}$ is equal to

$$\frac{(p^2q^2-p^2-q^2+1)}{16\cdot 36}(1-c)^2 + \frac{2p^2q^2 - pq(p^2+q^2) - (p-q)^2)}{48}(1-c) + \frac{(p^4+q^4-4p^3q-4pq^3+6p^2q^2)}{16}.$$

The first four parts of the lemma are clear, and the last two are straightforward calculations.

There is a second sesquilinear form on V defined by $\{v_\alpha, v_\beta\} = \delta_{\alpha\beta}$. If V_n is the subspace of V with basis $\{v_\alpha | \alpha = (k_1, \ldots, k_r) | \sum_{i=1}^r k_i = n\}$, then $V = \bigoplus_{n\geq 0} V_n$ and the spaces V_n are mutually orthogonal with respect to both forms. The first form is defined on V_n with respect to the second by a hermitian linear transformation $H_n = H_n(h,c) : \langle u, v\rangle_n = \{H_n u, v\}_n$. Let $P(n)$ be the dimension of V_n. It is the number of partitions of n. The Kac determinant formula (cf. [1]) is the key to the proof of Theorem FQS.

Kac determinant formula. *If $c = c(m)$ then*

$$\mathrm{det} H_n(h,c) = A_n \ \Pi_{k\leq n} \ \Pi_{pq=k} \ (h - h_{p,q})^{P(n-k)},$$

where A_n is a positive constant.

Lemma 3. *The form $\langle\cdot,\cdot\rangle_n$ is non-negative for $h \geq 0, c \geq 1$.*

Proof. By continuity it suffices to treat pairs for which $h > 0, c > 1$. Since the previous lemma implies that $\mathrm{det} H_n(h,c)$ is nowhere zero in this region, it suffices to prove that the form is positive for one pair (h,c). If $\alpha = (k_1, \ldots, k_r), r = r(\alpha), n(\alpha) = k_1 + \ldots + k_r$, set $v'_\alpha = L_{-k_r} \ldots L_{-k_1} v_\phi$. It is generally different than v_α. It clearly suffices to show that for a given c and h large,

$$\langle v'_\alpha, v'_\alpha\rangle = c_\alpha h^{r(\alpha)}(1 + o(1)), \ c_\alpha > 0 \tag{3.1}$$

$$\langle v'_\alpha, v'_\beta\rangle = o(h^{(r(\alpha)+r(\beta))/2}), \ \alpha \neq \beta. \tag{3.2}$$

This is proved by induction on $n(\alpha) + n(\beta)$. First of all $L_k^a L_{-k}^a$ is equal to

$$L_k^{a-1}(bL_0 + d)L_{-k}^{a-1} + L_k^{a-1} L_{-k} L_k L_{-k}^{a-1}, \ b > 0.$$

Moving the single L_k in the second term ever further to the right, we obtain finally $L_k^a L_{-k}^a = L_k^{a-1}(bL_0 + d)L_{-k}^{a-1} + L_k^{a-1} L_{-k}^a L_k, b > 0$. Take $k_1 \geq k_2 \geq \ldots \geq k_r > k$. If $\alpha = (k_1, \ldots, k_r, k, \ldots, k)$, then

$$\begin{aligned}\langle v'_\alpha, v'_\alpha\rangle &= \langle L_{k_1} \ldots L_{k_r} L_k^a L_{-k}^a L_{-k_r} \ldots L_{-k_1} v_\phi, v_\phi\rangle \\ &= c_{k,a} h(1 + o(h))\langle L_{k_1} \ldots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \ldots L_{-k_1} v_\phi, v_\phi\rangle \\ &+ \langle L_{k_1} \ldots L_{k_r} L_k^{a-1} L_{-k}^a L_k L_{-k_r} \ldots L_{-k_1} v_\phi, v_\phi\rangle\end{aligned}$$

with $c_{k,a} > 0$. In the second term we move the L_k further and further to the right obtaining the sum of

$$(k+k_r)\langle L_{k_1}\dots L_{k_r}L_k^{a-1}L_{-k}^aL_{-k_r}\dots L_{-k_{j+1}}L_{-(k_j-k)}L_{-k_{j-1}}\dots L_{-k_1}v_\phi,v_\phi\rangle.$$

The induction assumption together with the defining relations for $\mathfrak{v}$ implies readily that each of these terms is $o(h^{r(\alpha)})$ and that

$$\langle L_{k_1}\dots L_{k_r}L_k^{a-1}L_{-k}^{a-1}L_{-k_r}\dots L_{-k_1}v_\phi,v_\phi\rangle=\langle v'_\gamma,v'_\gamma\rangle=c_\gamma h^{r(\gamma)}(1+o(1)),$$

if $\gamma=(k_1,\dots,k_r,k,\dots,k)$, with k repeated $a-1$ times, so that $r(\alpha)=1+r(\gamma)$.

On the other hand, if $\beta=(ell_1,\dots,ell_s,k,\dots,k)$, with k repeated $a'\le a$ times, $a>0,a'\ge 0, ell_s\ge k$ even if $a'=0$, then

$$\langle v'_\beta,v'_\alpha\rangle=\langle L_{k_1}\dots L_{k_r}L_k^aL_{-k}^{a'}L_{-\,ell\,s}\dots L_{-\,ell_1}v_\phi,v_\phi\rangle$$

is equal to the sum of

$$c_{k,a'}h(1+o(1))\langle L_{k_1}\dots L_{k_r}L_k^{a-1}L_k^{a'-1}L_{-\,ell_s}\dots L_{-\,ell_1}v_\phi,v_\phi\rangle$$

and

$$\Sigma_j(k+\underset{j}{ell})\langle L_{k_1}\dots L_{k_r}L_k^{a-1}L_{-k}^{a'}L_{-\,ell_s}\dots L_{-\,ell_{j+1}}L_{-(ell_j\,-k)}L_{-\,ell_{j-1}}\dots L_{-\,ell_1}v_\phi,v_\phi\rangle.$$

We take $c_{k,0}=0$ if $a'=0$. So induction yields (3.2).

Observe that if $m>0$ and $p>q$ then $h_{p,q}>h_{q,p}$. If $h\ge 0$ and $m>0$ define $M>0$ by $M^2=1+4m(m+1)h$. Then $M\ge 1$. Let D be the closed shaded region in the diagram I. It is bounded by the lines

$$mx-(m+1)y=\pm M \text{ and } (m+1)x-my=M.$$

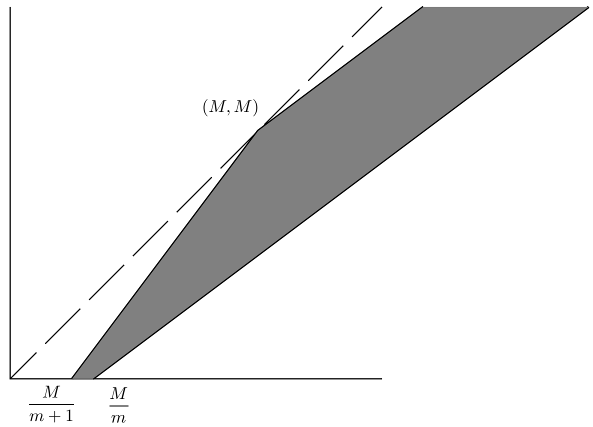

Diagram I

Lemma 4.

(a) $h_{p,q} \geq h \geq h_{q,p}$ *if and only if* $(p,q)\epsilon D$.

(b) D *contains an integral point* (p,q) *with* $q > 0$.

Proof. Since $h_{p,q} \geq h$ if and only if $((m+1)p - mq)^2 \geq M^2$ and $h \geq h_{q,p}$ if and only if $((m+1)q - mp)^2 \leq M^2$, the first statement of the lemma is clear. For the second choose a large integer p and let $a = \frac{mp-M}{m+1}$. Then the points (p,q) with $a \leq q \leq a + \frac{2M}{m+1}$ lie in D. So do the points $(p+1,q), a + \frac{m}{m+1} \leq q \leq a + \frac{m+2M}{m+1}$ and so on. So we need only show that one of the intervals $\left[a + \frac{km}{m+1}, a + \frac{km+2M}{m+1}\right], k \,\epsilon\, \mathbf{Z}, k \geq 0$, contains an integer. This is clear if $\frac{m}{m+1}$ is irrational. Otherwise, increasing q if necessary, we may suppose that a is as close to its integral part as any $a + \frac{km}{m+1}$. Then $a + \frac{m}{m+1} < [a]+1$, but $a + \frac{m+2M}{m+1} \geq a + \frac{m+2}{m+1} > [a]+1$, and the interval $\left[a + \frac{m}{m+1}, a + \frac{m+2M}{m+1}\right]$ contains $[a]+1$.

Let $p(h,c) = \min_{(p,q)\epsilon D} p$ and let $q(h,c) = \min_{(p,q)\epsilon D} q$. It is clear that

$$P(h,c) = (p(h,c), q(h,c))\epsilon D.$$

In the following geometrical arguments, it is sometimes necessary to recall that $h - h_{p_0,p_0} < 0$ if and only if $p_0 > M$.

Lemma 5. *If $P(h,c)$ lies in the interior of D then $\langle v,v\rangle$ assumes negative values in V.*

Proof. Let $(p,q)=P(h,c)$ and let $n=pq$. If $p_0q_0\le n, p_0\ge q_0$ and $(p_0,q_0)\ne(p,q)$ then either $p_0<p$ or $q_0<q$ so that $(p_0,q_0)\notin D$. In general set

$$\begin{aligned}\phi_{p_0,q_0}&=(h-h_{p_0,q_0}),\quad p_0\ne q_0,\\ &=h-h_{p_0,q_0},\quad p_0\ne q_0.\end{aligned}$$

If $(p_0,q_0)\notin D$ and $p_0\ne q_0$ then $\phi_{p_0,q_0}>0$.

Suppose that for some p_0 with $p_0^2\le pq$ we had $h-h_{p_0,p_0}<0$. Then there would be a minimum such p_0 and if $n_0=p_0^2$ then

$$\det H_{n_0}=A_{n_0}\prod_{\substack{p_1\ge q_1\\ n_1=p_1q_1\le n_0}}\phi_{p_1,q_1}^{P(n_0-n_1)}$$

Since $P(h,c)$ lies in the interior of $D, p\ne q$ and none of the pairs (p_1,q_1) that intervene here lie in D. Moreover, all terms of the products are positive save $\phi_{p_0,p_0}^{P(0)}=\phi_{p_0,p_0}$. Since this is negative, $\langle\cdot,\cdot\rangle$ assumes negative values on V_{n_0}.

If, however, $\phi_{p_0,p_0}>0$ for all $p_0\le q$ then the same argument shows that $\det H_n<0$, so that $\langle\cdot,\cdot\rangle$ assumes negative values on V_n.

The treatment of those points (h,c) for which $P(h,c)$ lies on the boundary of D is more delicate. There are at first three possibilities for $(p,q)=P(h,c)$:

(A) $mp-(m+1)q=M$;

(B) $(m+1)p-mq=M$;

(C) $mp-(m+1)q=-M, p\ne q$;

Lemma 6. *The case (C) above does not occur.*

Proof. It is clear from the diagram defining D that in case (C), $p\ge M$, $q\ge M$. If $q=1$ then $M=1$ and $p=1$, so that we have rather case (B). If $q>1$ then $p>1$ and $(m+1)(q-1)-m(p-1)=(m+1)q-mp-1$, so that $M>(m+1)(q-1)-m(p-1)>-M$. Moreover, $(m+1)(p-1)-m(q-1)-M=(m+1)(p-1-q)-m(q-1-p)=(2M+1)(p-q)-1$. Since $m\ge 2$ this is positive if $p\ne q$. Consequently $(p-1,q-1)\,\epsilon\, D$, and this is a contradiction.

Fix (p,q). In case (A) we have $h=h_{q,p}(m), c=c(m)$. In case (B) we have $h=h_{p,q}(m), c=c(m)$.

Lemma 7.

(a) *The set of all $m \geq 2$ for which $h = h_{q,p}(m), c = c(m)$ yields case (A) is the interval $m > q + p - 1$.*

(b) *The set of all $m \geq 2$ for which $h = h_{p,q}(m), c = c(m)$ yields case (B) is the interval $m > q+p-1$ if $(p, q) \neq (1, 1)$ and is the interval $m \geq 2$ if $(p, q) = (1, 1)$.*

It will be helpful, when proving this and the following lemmas, to keep the diagrams IIA and IIB in mind.

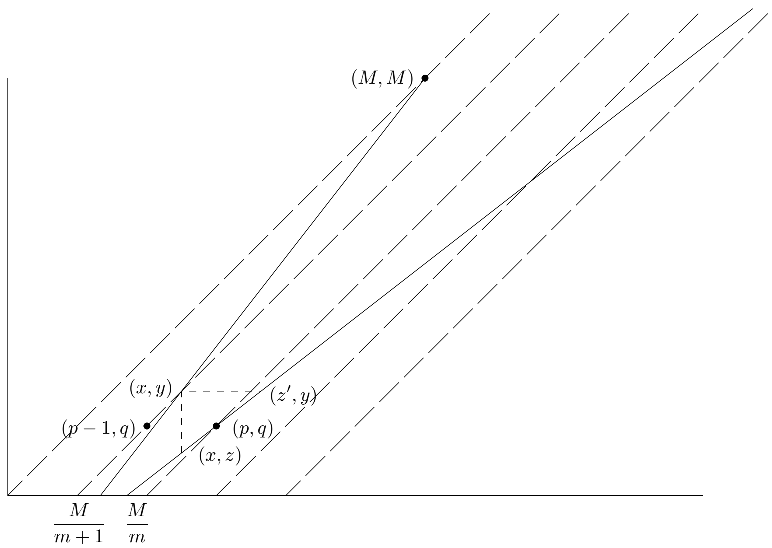

Diagram IIA

Proof. We first show that if $h_{q,p}(m_0), c(m_0)$ yield case (A) then so does $(h_{q,p}(m), c(m))$ for $m \geq m_0$. It is clear from the diagram that it is sufficient to verify that M, $\frac{M}{m+1}$, and $\frac{M}{m}$ are increasing functions of m. But $M = m(p-q) - q$, $\frac{M}{m} = (p-q) - \frac{q}{m}$, $\frac{M}{m+1} = (p-q) - \frac{p}{m+1}$. It is also clear that

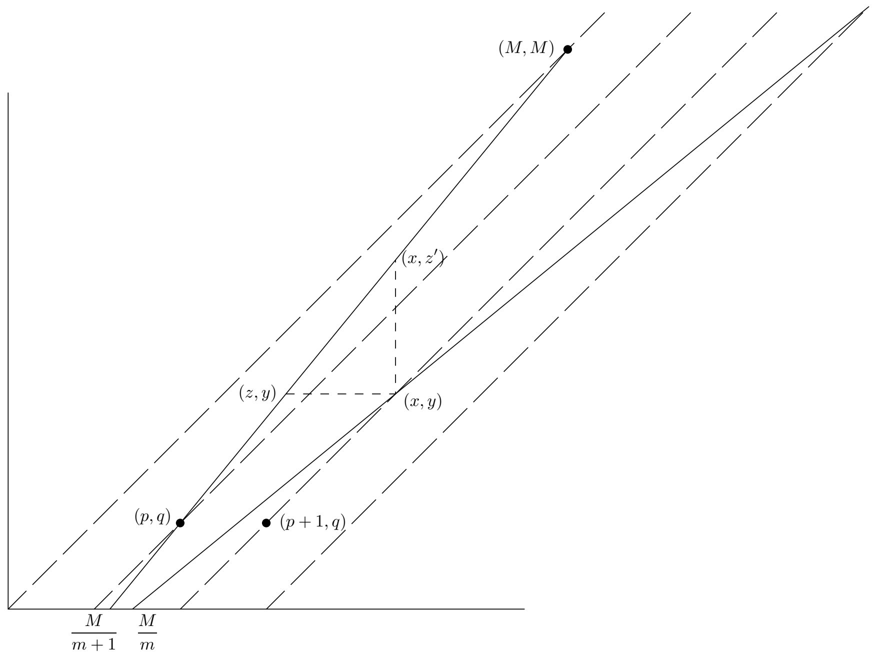

Diagram IIB

we can decrease m without passing out of case (A) so long as $M = m(p-q) - q$ remains greater than or equal to 1 and $(m+1)(p-1) - mq > mp - (m+1)q$. But

$$(m+1)(p-1) - mq = mp - (m+1)q \Leftrightarrow m = p + q - 1.$$

As we decrease to these points, M decreases to

$$(p+q-1)(p-q) - q = p^2 - q^2 - p = (p-1)^2 - q^2 + p - 1.$$

This number is greater than 1 because $p > q \geq 1$.

For case (B), $M = m(p-q) + p$ is a non-decreasing function of m, and $\frac{M}{m} = (p-q) + \frac{p}{m}, \frac{M}{m+1} = (p-q) + \frac{q}{m+1}$ are decreasing functions. Since the slope of $mp - (m+1)q = M$ is $1 - \frac{1}{m+1}$, it is increasing and the conclusion is the same. The minimal value of m is given by

$$(m+1)p - mq = mp - (m+1)(q-1) \Leftrightarrow m = p + q - 1.$$

because

$$(p+q-1)(p-q)+p=p^2-q^2+q\geq 1,$$

unless $p=q=1$ when m cannot go below 2.

In case (A) the intersection of the two lines $(m+1)x-my=M$ and $x-y=p-q-1$ is a point $(x(m),y(m))$ with $p'\geq x(m)>p'-1$ where p' is an integer, $p'\geq p$. If $x(m)=p'$ then $y(m)=q'=p'-p+q+1$, and $m=p'+q$. Thus $m\,\epsilon\,\{2,3,\ldots\}$, $p'<m, q'\leq p'$ and $c=c(m)$, $h=h_{p',q'}(m)$.

In case (B) the intersection of the lines $x-y=p-q+1$ and $mx-(m+1)y=M$ is a point $(x(m),y(m))$ with $p'\geq x(m)>p'-1, p'-1\geq p$. If $x(m)=p'$ then $m=p+q'$ lies in $\{2,3,\ldots\}, q\leq p, p<m$ and $c=c(m), h=h_{p,q}(m)$.

Thus to prove the theorem it suffices to establish the following proposition.

Proposition. *If case (A) or (B) obtains and $p'>x(m)>p'-1$ then the form $\langle\cdot,\cdot\rangle$ assumes negative values in V.*

We assume the contrary and derive a contradiction. We occasionally abbreviate $c(m)$ to c and $h_{q,p}(m)$ or $h_{p,q}(m)$ to $h(m)$ or to h.

Lemma 8.

(a) *Suppose $p'>x(m)>p'-1$. If (p_1,q_1) lies on the boundary of $D(h,c)$ and $p_1q_1\leq p'q'$ then $(p_1,q_1)=(p,q)$.*

(b) *Define m' by $p'=x(m')$ and set $c'=c(m')$, $h'=h_{q,p}(m')$ or $h_{p,q}(m')$. If (p_1,q_1) lies on the boundary of $D(h',c')$ and $p_1q_1\leq p'q'$ then (p_1,q_1) is (p,q) or (p',q').*

Proof. Set $(x,y)=(x(m),y(m))$ and define z,z' as indicated by the diagrams. It clearly suffices to show that in case (A) $y-z<2, z'-x<2$, and that in case (B), $x-z<2, z'-y<2$. In case (A) elementary algebra yields $m=x+q, y-z=\frac{x+z}{m}=1+\frac{z-q}{x+q}$ and $\frac{z-q}{x+q}=\frac{x-p}{x+q}\cdot\frac{z-q}{x-p}<1$. On the other hand $z'-x=\frac{x+y}{m}=1+\frac{y-q}{p+y-1}<2$. A similar argument works for case (B).

Since p,q and p' are fixed it will be useful to let C denote the curve $c=c(m)$, $h=h_{q,p}(m)$ (A) or $h=h_{p,q}(m)$ (B), $m>p'-1$.

Lemma 9.

(a) If $x(m) > p' - 1$, $x(m) \neq p'$, and $n_1 \leq n'$, then the dimension of the space of null vectors in V_{n_1} is $P(n_1 - n)$.

(b) If $x(m) = p'$ and $n_1 < n'$ then the dimension of the space of null vectors in V_{n_1} is $P(n_1 - n)$, but if $n_1 = n'$ it is $P(n_1 - n) + 1$.

Proof. Observe that $P(n_1 - n) = 0$ if $n_1 < n$ and that when this is so the lemma is clear. So take $n_1 \geq n$ and denote the pertinent dimension by $d^0_{n_1}$. We begin by showing that $d^0_{n_1} > 0$ and that $d^0_{n_1} \leq P(n_1 - n)$ unless $x(m) = p'$ and $n_1 = n'$ when $d^0_{n_1} \leq P(n_1 - n) + 1$.

For $0 \leq c < 1$, m is locally an analytic function of c and we may write $h_{p,q}(m) = h_{p,q}(c) = h(c)$ or $h_{q,p}(m) = h_{q,p}(c) = h(c)$. Fix c and consider $H_{n_1}(h, c)$ as a function of h near $h(c)$. Its eigenvalues are the roots of a polynomial equation with real analytic, indeed polynomial, coefficients and they are all real for h real. It is easily seen that this implies that there is no ramification at $h = h(c)$ and that in a neighborhood of this point there are expansions

$$\alpha_i(h) = \alpha_{i0} + \alpha_{i1}(h - h(c)) + \alpha_{i2}(h - h(c))^2 + \ldots,\ 1 \leq i \leq P(n_1)$$

for the eigenvalues of H_{n_1}. Thus

$$\det H_{n_1}(h, c) = \Pi_{i=1}^{P(n_1)}(\alpha_{i0} + \alpha_{i1}(h - h(c)) + \ldots),$$

and the power of $h - h(c)$ that divides it is greater than or equal to the number of zero eigenvalues of $H_{n_1}(h(c), c)$. On the other hand, the left side is equal to

$$A_n \Pi_{k \leq n_1} \Pi_{p_1 q_1 = k}(h - h_{p_1, q_1}(c))^{P(n_1 - k)},$$

and $h_{p_1,q_1}(c) = h(c)$ only if (p_1, q_1) or (q_1, p_1) lies in boundary of D. Thus the assertion follows from Lemma 8.

Choosing $n_1 = n$, we see in particular that the dimension of the null space of V_n is 1. Thus if $m > p' - 1$ then in a neighborhood of $(h(m), c(m))$ we can find an analytic function $v(h, c)$ with values in V_n such that $v(h, c)$ has length 1, is an eigenvector of $H_n(h, c)$, and corresponds to the eigenvalue 0 when (h, c) falls on the curve C.

Since

$$L_0 v(h(m), c(m)) = (h(m)+n)v(h(m), c(m)),$$

$$L_k v(h(m), c(m)) = 0, \quad k > 0,$$

there is a homomorphism of $\mathfrak{v}$-modules, ϕ: $V^{h(m)+n,c(m)} \to V^{h(m),c(m)}$, taking $v_\phi^{h(m)+n,c(m)}$ to $v(h(m), c(m))$. If it is injective on $V_{n_1-n}^{h(m)+n,c(m)}$ then $d_{n_1}^0 \geq P(n_1 - n)$ because the image consists of null vectors. Since $d_{n_1}^0$ is lower semicontinuous, $d_{n_1}^0$ will be greater than or equal to $P(n_1 - n)$ everywhere on C if it is so on a dense set. The homomorphism ϕ will be injective if $\det H_{n_1-n}^{h(m)+n,c(m)} \neq 0$ because the kernel consists of null vectors. So it is enough to show that this determinant does not vanish identically on C. However, if $h(m)+n = h_{p_1,q_1}(m)$ then

$$((m+1)p + mq)^2 = ((m+1)p_1 - mq_1)^2$$

or

$$(mp + (m+1)q)^2 = ((m+1)p_1 - mq_1)^2.$$

This can occur for at most two values of m.

It remains to show that at m' the dimension of the space of null vectors in $V_{n'}$ is $P(n'-n)+1$. For this we need further lemmas.

Lemma 10. $\det H_{n'-n}^{h(m')+n,c(m')} \neq 0$.

Proof. It has to be shown that equality $h(m')+n = h_{p_1,q_1}(m')$, $p_1 q_1 \leq n'-n$ is impossible. This equality amounts to

$$(m'p + (m'+1)q)^2 = ((m'+1)p_1 - m'q_1)^2 \tag{A}$$

or

$$((m'+1)p + m'q)^2 = ((m'+1)p_1 - m'q_1)^2. \tag{B}$$

It is not supposed that $p_1 \geq q_1$.

The first equation implies that $m'p + (m'+1)q = \pm((m'+1)p_1 - m'q_1)$ or $m'(p \pm q_1) = (m'+1)(\pm p_1 - q)$. Since m' is an integer this implies $(p \pm q_1) = a(m'+1), (\pm p_1 - q) = am'$. Since $n' = p'q' = (m'-q)(m'-p+1)$ the inequality $n' \geq n + p_1 q_1$ becomes

$$(m'-q)(m'-p+1) \geq a(m'+1)q - am'p + a^2 m'(m'+1)$$

or

$$((1+a)(m'+1)-p)((1-a)m'-q) \geq 0.$$

Since $m' = p' + q = p + q' - 1, m' > q, m' + 1 > p$. So the inequality is possible only for $a = 0$, but a cannot be 0. The case (B) is treated in a similar fashion.

For $n_1 < n'$ or $m \neq m'$ we let $U_{n_1} = U_{n_1}(m)$ be the space of null vectors in V_{n_1}. For h, c close to $h(m'), c(m')$ we let $U_{n_1}(h,c)$ be the span of

$$\{L_{-k_1} \dots L_{-k_r} v(h,c) | k_1 \geq \dots \geq k_r > 0, \Sigma k_i = n' - n\}.$$

We set $U_{n'}(m) = U_{n'}(h(m), c(m))$, the two definitions of $U_{n'}(m)$ coinciding when they both apply. Thus for $m > p' - 1$, $U_{n_1}(m)$ is defined and analytic as a function of m. Let W_{n_1} be its orthogonal complement with respect to the form $\{\cdot,\cdot\}$. It follows from that part of Lemma 9 already proved that the restriction $J_{n_1} = J_{n_1}(m)$ of H_{n_1} to W_{n_1} is non-singular unless $n_1 = n', m = m'$. In particular, our assumption, which was made for a particular m, implies that $J_{n_1}(m)$ is positive for all $m > p' - 1$ if $n_1 < n'$.

Lemma 11. *Near* m', $\det J_{n'}(m) = \delta(m)(m - m')$ *where* $\frac{1}{\delta} \geq |\delta(m)| \geq \delta > 0$.

It will follow from this lemma that the remaining assertion of Lemma 9 is true. In addition the lemma together with our assumption on the non-negativity of $\langle \cdot,\cdot \rangle$ for a particular m, $p' > x(m) > p' - 1$, will imply that the form takes negative values for $m > m'$ because $\det J_{n'}(m)$ changes sign at m'.

Let $v(h,c)$, defined in a neighborhood of $(h(m'), c(m'))$, correspond to the eigenvalue $\alpha(h,c)$ of $H_n(h,c)$. All the other eigenvalues of $H_n(h,c)$ are bounded above and, if the neighborhood is sufficiently small, away from 0. On the other hand, all factors $h - h_{p_1,q_1}(c) = h - h_{p_1,q_1}(m)$, $c = c(m)$, of $\det H_n(h,c)$ are bounded away from 0 in a neighborhood of $h(m')$, $c(m')$ except for $h - h(c)$, where

$h(c)$ is $h_{q,p}(c)$ or $h_{p,q}(c)$ according as we are dealing with case A or case B. Thus we have the following lemma.

Lemma 12. *In a neighborhood of $(h(m'),c(m'))$ we have $\alpha(h,c)=a(h,c)(h-h(c))$ with $\frac{1}{a}\geq|a(h,c)|\geq a>0$, a being a constant.*

Here $h(c)$ is $h_{q,p}(m)$ (A) or $h_{p,q}(m)$ (B), $c=c(m)$. More generally we have

Lemma 13. *Let $K_{n'}(h,c)$ be the restriction of $H_{n'}(h,c)$ to $U_{n'}(h,c)$. Then, in a neighborhood of $(h(m'),c(m'))$, $\det K_{n'}(h,c)=k(h,c)\alpha(h,c)^{P(n'-n)}$, with $\frac{1}{k}\geq|k(h,c)|\geq k>0$.*

Proof. The deteriminant of $K_{n'}(h,c)$ is that of the form $\langle\cdot,\cdot\rangle_{n'}$, calculated with respect to a basis of $U_{n'}(h,c)$ orthogonal with respect to the form $\{\cdot,\cdot\}_n$. However the basis $\{\phi(v_\alpha)|v_\alpha\,\epsilon\,V^{h(m)+n,c(m)},n(\alpha)=n'-n\}$ is related to such a basis by a matrix whose determinant is bounded in absolute value above and below. So it is enough to consider $\det(\{\phi(v_\alpha),\phi(v_\beta)\})$.

We have

$$\begin{aligned}\langle\phi(v_\alpha),\phi(v_\beta)\rangle&=\langle L_{ell_s}\ldots L_{ell_1}L_{-k_1}\ldots L_{-k_r}v(h,c),v(h,c)\rangle\\&=\{L_{ell_s}\ldots L_{ell_1}L_{-k_1}\ldots L_{-k_r}v(h,c),H_{n'}(h,c)v(h,c)\}\\&=\alpha(h,c)\{L_{ell_s}\ldots L_{ell_1}L_{-k_1}\ldots L_{-k_r}v(h,c),v(h,c)\}.\end{aligned}$$

At $h(m),c(m)$ the value of $\det(\{L_{ell_s}\ldots L_{ell_1}L_{-k_1}\ldots L_{-k_r}v(h,c),v(h,c)\})$ is

$$\det(\langle v_\alpha,v_\beta\rangle_{n'-n}^{h(m')+n,c(m')}).$$

By Lemma 10 this is not 0. Lemma 13 follows.

In a neighborhood of $h(m),c(m)$ we decompose $V_{n'}$ as an orthogonal sum $U_{n'}\oplus W_{n'}$. The linear transformation $H_{n'}(h,c)$, or its matrix with respect to a compatible basis, then decomposes into blocks. I claim that the entries in the off-diagonal blocks are $O(h-h_{p,q}(c))$ in a neighborhood of $h(m),c(m)$. To verify this it is sufficient, for the pertinent basis can be supposed to depend analytically on h,c, to verify that they are zero when $h=h_{p,q}(c)$, but that is clear by the definition of $U_{n'}$.

It follows that

$$\det H_{n'}(h,c) = \det J_{n'}(h,c)\det K_{n'}(h,c) + O((h-h_{p,q}(c))^{P(n'-n)+1}) \qquad (1)$$

if $J_{n'}(h,c)$ is the matrix in the diagonal block corresponding to $W_{n'}$. Since

$$\det H_{n'}(h,c) = A_{n'}\Pi_{k\leq n'}\Pi_{p_1q_1=k}(h-h_{p_1q_1}(c))^{P(n'-p_1q_1)}$$

we may divide the relation (1) by $(h-h_{p_1q_1}(c))^{P(n'-n)}$ and then set $h = h_{p,q}(c), c = c(m)$. The result clearly yields Lemma 11 because $h(m') = h_{p_1,q_1}(m'), p_1, q_1 \leq n'$, only if (p_1,q_1) is (q,p) or (p',q') (case A) or (p,q) or (q',p') (case B).

Our assumption that $H_{n_1}(h(m),c(m))$ is non-negative for a given $m, p' > m > p'-1$, has led to the conclusion that $J_{n_1}(m)$ is positive for large m and $n_1 < n'$ but that $J_{n'}(m)$ has negative eigenvalues for large m. We show not that this is impossible.

As m approaches infinity, the point $(h(m),c(m))$ approaches $(h_0,c_0) = \left(\frac{(p-q)^2}{4},1\right)$. If $p \neq q$ a suitable coordinate on the curve is $\mu = \frac{1}{m}$. If $p = q$ we may take $\mu = 1-c$. All the matrices $H_{n_1}(\mu) = H_{n_1}(m) = H_{n_1}(h(m),c(m))$ are analytic functions of μ. The eigenvalues of $H_{n_1}(\mu)$ are given by power series.

$$\alpha_i = \alpha_i(\mu) = \alpha_{i0} + \alpha_{i1}\mu + \alpha_{i2}\mu^2 + \ldots$$

Let $V^1_{n_1}(\mu)$ be the space spanned by the eigenvectors corresponding to α_i with $\alpha_{i0} = 0$; let $V^2_{n_1}(\mu)$ be the space spanned by the eigenvectors corresponding to α_i with $\alpha_{i0} = \alpha_{i1} = 0$ and so on. One proves by induction that these spaces are well defined, depend analytically on μ (in the sense that we have analytic functions $v_1(\mu),\ldots,v_{P(n_1)}(\mu)$, such that $\{v_1(\mu),\ldots,v_{d_k}(\mu)\}$, $d_k = \dim V^k_{n_1}$ forms a basis of $V^k_{n_1}(\mu)$ for each μ), and that $\mu^{-k}\{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\}, i \leq d_k, j \leq P(n_1)$ is analytic for small μ. It can even be supposed that $\{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\} = 0, i \leq d_k, j > d_k$.

Let $V^k = \oplus_{n_1}V^k_{n_1}(0)$ and $X^k = V^k/V^{k+1} = \oplus_{n_1}V^k_{n_1}(0)/V^{k+1}_{n_1}(0)$. If $u = \Sigma_{i\leq d_k}a_iv_i(0)\,\epsilon\, V^k_{n_1}(0)$ and $v = \Sigma_{i\leq d_k}b_iv_i(0)\,\epsilon\, V^k_{n_2}(0)$, define $\langle u,v\rangle^{(k)}$ to be 0 if $n_1 \neq n_2$, and if $n_1 = n_2$ set

$$\begin{aligned}\langle u,v\rangle^{(k)} = \langle u,v\rangle^{(k)}_{n_1} &= \Sigma a_i\bar{b_j}\lim_{\mu\to 0}\mu^{-k}\langle v_i(\mu), v_j(\mu)\rangle \\ &= \Sigma a_i\bar{b_j}\lim_{\mu\to 0}\{\mu^{-k}H_{n_1}(\mu)v_i(\mu), v_j(\mu)\}.\end{aligned}$$

It is clear that $H_{n_1}(\mu)$ is non-negative for small μ if and only if the forms $\langle u,v\rangle^{(k)}_{n_1}$ are all positive.

Lemma 14.

(a) The spaces V^k are all invariant under $\pi = \pi^{h_0,c_0}$, so that $\mathfrak{v}$ operates on X^k.

(b) The form $< \cdot, \cdot >^{(k)}$ on X^k satisfies $< L_m x, y >=< x, L_{-m} y >, m \,\epsilon\, \mathbf{Z}$.

Proof. Set $L_m(\mu) = \pi^{h(\mu),c(\mu)}(L_m)$ and $L_m = L_m(0)$. We have to show for each n_1 that $L_m v_i \epsilon V^k$ if $v_i = v_i(0)$ and $i \leq d_k$. However

$$L_m v_i = \lim_{\mu \to 0} L_m(\mu) v_i(\mu) = \lim \Sigma_j c_{ij}(\mu) v'_j(\mu)$$

where the c_{ij} are analytic functions of μ. It is to be shown that $c_{ij}(0) = 0$ for $j > d'_k$. The primes refer to $n_2 = n_1 - m$ rather than to n_1. In other words it has to be shown that $\{H_{n_2}(\mu) L_m(\mu) v_i(\mu), v'_{ell}(\mu)\} = O(u^k)$ for all ell. Since $H_{n_2}(\mu) L_m(\mu) = L^*_{-m}(\mu) H_{n_1}(\mu)$, the adjoint of $L_{-m}(\mu)$ being taken with respect to the form $\{\cdot, \cdot\}$, this is clear. So is the second assertion of the lemma.

For any $h \geq 0$ the representation $\pi^{h,1}$ on $V^{h,1}$ has a unique irreducible quotient $\rho^{h,1}$ on $X^{h,1}$, which by Lemma 3 carries a hermitian form for which $\rho^{h,1}$ is unitary in the sense that the adjoint $\rho^{h,1}(L_m)$ is $\rho^{h,1}(L_{-m})$. Such a form is unique up to a scalar multiple. Take in particular $h = \frac{r^2}{4}, r \,\epsilon\, \mathbf{Z}$. Then $h = h_{p_2,q_2}(c)$ if and only if $(p_2 - q_2)^2 = r^2$. In particular, $h = h_{r+1,1}(c)$. Thus the lowest weight for a null vector in V is $r+1$ and $h + r + 1 = \frac{(r+2)^2}{4}$, so that the kernel of $V^{h,1} \to X^{h,1}$ contains a quotient of $V^{h',1}, h' = \frac{(r+2)^2}{4}$. Thus $V^{h,1}$ admits a sequence of invariant subspaces $V^{h,1} = V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq V^{h,1}(2)$ such that the representation on $V^{h,1}(0)/V^{h,1}(1)$ is $\rho^{h,1}$ and that on $V^{h,1}(1)/V^{h,1}(2)$ is $\rho^{h',1}$. In general set $h^{(ell)} = \frac{1}{4}(r + 2\,ell)^2$.

Lemma 15. *$V^{h,1}$ admits an infinite decomposition series $V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq \ldots \supseteq V^{h,1}(ell) \supseteq \ldots$ such that the representation on the quotient $V^{h,1}(ell)/V^{h,1}(ell+1)$ is $\rho^{h(ell),1}$.*

Proof. If $\lambda = h + k, k \,\epsilon\, \mathbf{Z}, k \geq 0$, let $d_\lambda = \dim\{v \epsilon V^{h,1} | L_0 v = \lambda v\}, d_\lambda(ell) = \dim\{v \epsilon X^{h(ell),1} | L_0 v = \lambda v\}$. The lemma follows easily from a formula of Kac([2], Th. 5), according to which $d_\lambda = \Sigma_{ell=0}^{\infty} d_\lambda(ell)$. Indeed, suppose we have constructed an initial segment of the series $V^{h,1}(0) \supset \ldots \supset V^{h,1}(ell)$. Then $\frac{1}{4}(r + 2\,ell)^2$ is a lowest weight in $V^{h,1}(ell)$ and $\dim\{v \epsilon V^{h,1}(ell) | L_0 v = \frac{1}{4}(r + 2\,ell)^2\} = 1$. Take $V^{h,1}(ell+1)$ to be the sum of all invariant subspaces of $V^{h,1}(ell)$ for which the lowest weight is greater than $\frac{1}{4}(r + 2\,ell)^2$.

Now take $r = p - q$. It follows immediately from the preceding lemma that X^k is the direct sum of irreducible invariant subspaces X_j^k carrying distinct representations and that the restriction of $\langle \cdot, \cdot \rangle^k$ to X_j^k is either positive or negative. The assumption that we are trying to contradict implies that the form is positive if X_j^k contains non-zero vectors of weight $h + n_1, n_1 < n'$, but that for some j and k for which X_j^k contains vectors of weight $h + n'$, it is negative.

Thus the following lemma completes the proof of Theorem FQS.

Lemma 16. *The equation* $\frac{r^2}{4} + n' = \frac{1}{4}(r + 2\,ell)^2$ *has no solution* $ell \geq 0$ *in* $\mathbf{Z}$.

Proof. The equation may be written as $n' = ell(ell + r)$. Recall that n' is $(p+a)(q+a+1)$ in case A and $(p+a+1)(q+a)$ in case B, with $a \geq 0$. Since $r = p - q$, the equation is $(p+a+ell)(q+a+1-ell) = ell$ or $(p + a + 1 - ell)(q + a - ell) = -\,ell$. Both equations are manifestly impossible.

References

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