Rank-One Residues of Eisenstein Series*

Dedicated to Ilya Piatetski-Shapiro with admiration and affection

Introduction

The trace-class problem was for some years a rather embarrassing unsolved problem in the theory of automorphic forms. Although the absence of a solution did not seriously obstruct progress, it did undesirably, and in the general view unnecessarily, complicate the statements of various results, especially in the context of the trace formula. Fortunately, it has recently been solved by W. Müller in [TC].

If G is a reductive Lie group and Γ an algebraic subgroup then we may introduce the space $L^2(\Gamma\backslash G)$, on which G acts by right translation, and then the sum $L^2_d(\Gamma\backslash G)$ of all irreducible invariant subspaces of $L^2(\Gamma\backslash G)$. Let R_d be the representation of G on $L^2_d(\Gamma\backslash G)$. A weak form of Müller's result is that if f is a smooth compactly supported function on G then $R_d(f)$ is of trace class.

The proof proceeds in two steps. The theory of Eisenstein series provides a decomposition of $L^2_d(\Gamma\backslash G)$ into subspaces indexed by associate classes of parabolic subgroups. Thus the trace class problem reduces to a collection of problems, one for each class $\{P\}$ of associate parabolic subgroups. For the class G the problem is easy. The first step in Müller's analysis is to solve it for classes of maximal proper parabolic subgroups using a variant of an argument that H. Donnelly [EE] had used in the special case that the rank of Γ is one. The second step is to deduce the result in general from this special case by exploiting the construction within the theory of Eisenstein series of the entire discrete spectrum by iterated residues.

Donnelly's methods are based on classical techniques of the spectral theory of differential operators. In this paper, we offer an alternative approach to the first step; it combines some of J. Arthur's basic early results on the trace formula, but none of the difficult later work, with a simple estimate based on a differential inequality. Although the argument is relatively old, dating to discussions with Arthur in September of 1983, it is not long and is perhaps more in the spirit of the rest of Müller's proof. As it stands, the precise result on the estimation of eigenvalues is weaker than his, for the reason indicated in the Appendix, but it is not clear whether this will ultimately matter. Since I begin with Arthur's papers, I have to restrict myself to adele groups, or, if one prefers, to congruence subgroups, but the method is, in fact, completely general. To simplify the notation, I take the group G to be semisimple.

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Elements of the trace formula

The trace formula as it appears in the papers of Arthur is an equality between two infinite sums, the terms of each sum undergoing several avatars as the formula is repeatedly refined for applications. For the present purposes, we need only one side of the formula, the χ-expansion. Although our technique is inspired by the fine expansion of the papers [FD I] and [FD II], we need very little of the difficult analysis of these two papers and the papers [PW] and [IP] that preceded them; so we begin with a review of the course χ-expansion of [TF II], and then in the following section treat more fully those parts of the fine expansion that we need.

Recall ([TF II], §3) that the parameters χ that occur in the χ-expansions are classes of pairs, (M,ρ), where M is the Levi factor of a parabolic subgroup over $\mathbb{Q}$ and ρ is a cuspidal automorphic representation of $M(\mathbb{A})^1$. The course χ-expansion is a sum,

$$\sum_\chi J_\chi^T(f), \tag{1.1}$$

where

$$J_\chi^T(f) = \sum_P \frac{1}{n(A)} \int_{\Pi^G(M)} \mathrm{tr}(M_P^T(\pi)_\chi \cdot I_P(\pi,f)_\chi)\, d\pi. \tag{1.2}$$

The notation is taken from §1 and §3 of [TF I] and §3 of [TF II], where the expansion appears as Theorem 3.2, so that the sum is over standard parabolic subgroups P containing one of the Levi factors in χ. As in the final paragraph of §1 of [TF I] the symbol A denotes a split component of P; the integer $n(A)$ is also defined in that paragraph as the number of chambers in the Lie algebra of A.

The operators $M_P^T(\pi)_\chi$ will be described precisely later, at least for the cases of concern to us. What has to be stressed now is that the double sum over χ and P obtained upon substitution of (1.2) in (1.1) continues to converge if the integrand is replaced by a trace-class norm,

$$\|M_P^T(\pi)_\chi \cdot I_P(\pi,f)_\chi\|_1. \tag{1.3}$$

The operator $I_P(\pi,f)_\chi$ is introduced in §3 of [TF I] and is the operator associated to the function f in the induced representation on $\mathcal{H}_P(\pi)_\chi$. If $X_1,\dots,X_d$ is a basis for the Lie algebra of $G(\mathbb{R})$ orthonormal with respect to a positive form, Q, invariant under the given, fixed maximal compact subgroup, K, of $G(\mathbb{R})$, if D is a positive constant, and if

$$\Delta_1 = D - \sum_{i=1}^d X_i^2,$$

then the operators $I_P(\pi,\Delta_1^{-n})_\chi$ are defined for any positive integer n, and the double sum in (1.3) continues to converge when f is replaced by the product of Δ_1^{-n} with the characteristic function of an open compact subgroup of $G(\mathbb{A}_f)$, provided that n is sufficiently large.

Since $(\Delta_1\phi,\phi) \geq D(\phi,\phi)$ for all ϕ, the definition of the operator Δ_1^{-n} presents no problems. The smaller n can be taken, the better are the estimates. I take the n determined by the procedure of [TF I] and [TF II] that is reviewed in the Appendix, although the estimates are then quite poor in comparison with those obtained from other techniques (§3 of [TC]). This is a serious defect, but I make no attempt to remedy it. The number D is so chosen that $D - Q(\rho_P) > 0$ for all P, the element ρ_P being defined in the usual way (*cf.* p. 918 of [TF I]).

Whatever n is, denote Δ_1^{-n} by Δ. The assertion with which we begin is that

$$\sum_{\chi} \sum_{P} \frac{1}{n(A)} \int_{\Pi(M)} \|M_P^T(\pi)_{\chi,K_0} \cdot I_P(\pi,\Delta)_{\chi,K_0}\|_1 d\pi \tag{1.4}$$

is finite. This is Theorem 3.1 of [TF II]. Since K_0 will be fixed we drop it from the notation. Moreover, χ will also often be fixed, and then it too will be dropped from the notation, to appear again when necessary.

The quadratic form, Q, appearing in the definition of Δ can be so chosen that the operator $I_P(\pi,\Delta)$ acts as a scalar on isotypical subspaces for the maximal compact subgroup K. Since these subspaces are invariant under the operator $M_P^T(\pi)$, the product that appears in (1.4) is then easy to deal with. Indeed, various minor points will be easier to deal with if we take, as is customary, the form to be the negative of the Killing form on the Lie algebra of K and to be the Killing form itself on the orthogonal complement of that algebra.

Only those χ for which the Levi factor is that of a maximal parabolic subgroup will be of concern to us, and for these the operator $M_P^T(\pi)$ (the subscripts have been suppressed) can be calculated rather simply in terms of the intertwining operators appearing in the functional equation of the Eisenstein series.

Suppose that P is a maximal (proper) parabolic subgroup whose Levi factor appears in one of the pairs defining χ. As Lemma 4.1 of [TF II] makes clear, the truncation operator of Arthur applied to Eisenstein series associated to cusp forms on the Levi factors of maximal parabolic subgroups is the truncation denoted by a double prime in [ES]. The general inner-product formula of §4 of [TF II] reduces to the formulas on p. 135 of [ES].

If we denote the Eisenstein series $E(x,\phi_\zeta)$ of §3 of [TF I] by $E(x,\phi,\zeta)$ then the inner-product formula for

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \Lambda^T E(x,\phi,\lambda)\,\overline{E(x,\psi,\mu)}\,dx \tag{1.5}$$

is, for λ unequal to $\pm\bar{\mu}$, the sum of

$$\frac{1}{\langle \lambda+\bar{\mu},\beta\rangle}\left\{e^{\langle\lambda+\bar{\mu},T\rangle}(\phi,\psi) - e^{-\langle\lambda+\bar{\mu},T\rangle}(M(\lambda)\phi, M(\mu)\psi)\right\}$$

and

$$\frac{1}{\langle \lambda-\bar{\mu},\beta\rangle}\left\{e^{\langle\lambda-\bar{\mu},T\rangle}(\phi, M(\mu)\psi) - e^{-\langle\lambda-\bar{\mu},T\rangle}(M(\lambda)\phi, \psi)\right\}.$$

The notation is an obvious mixture of that of [ES] and that of Lemma 4.2 of [TF II]. Moreover, β has been so chosen that the volume that otherwise appears as a factor in the formula of that lemma disappears. Since the parabolic subgroup P is maximal, we may, and shall, treat λ, μ, and T as real or complex numbers.

If μ is taken equal to $\lambda = \sigma + i\tau$ the formula becomes the sum of

$$\frac{1}{2\sigma}\left\{e^{2\sigma T}(\phi,\psi) - e^{-2\sigma T}(M(\lambda)\phi, M(\lambda)\psi)\right\}$$

and

$$\frac{1}{2i\tau}\left\{e^{2i\tau T}(\phi, M(\lambda)\psi) - e^{-2i\tau T}(M(\lambda)\phi,\psi)\right\}.$$

If σ is taken to be 0 so that λ is purely imaginary the second term remains meaningful provided that τ is not 0. Taking the limit of the first term as σ approaches 0, we see that it becomes

$$i(\Theta^{-1}(\tau)\Theta'(\tau)\phi,\psi) + 2T(\phi,\psi),$$

if $\Theta(\tau)$ is the unitary operator $M(i\tau)$ and the prime denotes differentiation with respect to the real variable τ.

It is, however, better to take

$$\Theta(\tau) = e^{-2i\tau T}M(i\tau),$$

for then the inner-product formula becomes

$$-\left\{\frac{1}{i}(\Theta^{-1}(\tau)\Theta'(\tau)\phi,\psi) + \frac{1}{2i\tau}(\Theta(\tau)\phi,\psi) - (\phi,\Theta(\tau)\psi)\right\}. \tag{1.6}$$

(Observe that there seems to be an error of sign in the corresponding formula on p. 145 of [ES]. It does not affect the argument.)

Since $\Theta(\tau)$ is a unitary operator it is given by

$$\Theta(\tau) = e^{i\theta(\tau)},$$

with $\theta(\tau)$ hermitian. If we make the dependence of Θ or of θ on π and χ explicit, we obtain the formula

$$M_P^T(\pi,\tau) = M_P^T(\pi_{i\tau})_\chi = -\frac{1}{i}\Theta_\pi^{-1}(\tau)_\chi\Theta_\pi'(\tau)_\chi - \frac{1}{\tau}\sin(\theta_\pi(\tau)_\chi), \tag{1.7}$$

but it will seldom be necessary to deal with such an accumulation of subscripts. Observe that the measure $d\pi$ that appears in (1.4) is, apart from a constant that has no importance here, simply $d\tau$.

We now have the first fact that we need. It is a reformulation of the convergence of (1.4), but in a weakened form, since the sum is taken over standard maximal parabolic subgroups and over cuspidal representations π of $M(\mathbb{A})$ modulo the action of $i\mathfrak{a}^*$. As in §3 of [TF I] we denote an orbit of $i\mathfrak{a}^*$ by $\Pi^P(M)$, but to conform with [ES], from which the formula for (1.5) is taken, it is understood that we choose in each orbit the representative π that is trivial on the connected component of $A(\mathbb{R})$. Set

$$I_P(\pi_{i\tau},\Delta) = I_P(\pi,\tau,\Delta).$$

Assertion A. *The sum*

$$\sum_\chi\sum_P\frac{1}{n(A)}\sum_{\Pi^P(M)}\int_{-\infty}^{\infty}\|\left\{\frac{1}{i}\Theta_\pi^{-1}(\tau)_\chi\Theta_\pi'(\tau)_\chi + \frac{1}{\tau}sin(\theta_\pi(\tau)_\chi)\right\}\cdot I_P(\pi,\Delta)\|_1\ d\tau$$

is finite, as is

$$\sum_{\chi} \sum_{\Pi(G)} \|M_G^T(\pi)_\chi \cdot \pi(\Delta)_\chi\|_1 \,.$$

Observe that $I_P(\pi)$ is simply π when π is a representation of $G(\mathbb{A})$ itself. If $P = G$ and π is cuspidal then $M_G^T(\pi)$ is the identity. If π is a representation on square-integrable but noncuspidal automorphic forms, $M_P^T(\pi)$ is also readily calculated.

Suppose that Φ is the residue of $E(\cdot, \phi, \cdot)$ at λ and Ψ the residue of $E(\cdot, \psi, \cdot)$ at μ. Taking the residue of the formula for (1.5) at λ and then the residue of the result, treated of course as a function of the complex-conjugate variable, we obtain first

$$-\frac{1}{\langle \lambda + \bar{\mu}, \beta \rangle} e^{\langle \lambda + \mu, T \rangle} (m(\lambda)\phi, M(\mu)\psi) - \frac{1}{\langle \lambda - \bar{\mu}, \beta \rangle} e^{\langle \lambda - \bar{\mu}, T \rangle} (m(\lambda)\phi, \psi),$$

and then, recalling that λ and μ are necessarily real and positive, our second assertion.

Assertion B. *The integral*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \Lambda^T \Phi(x) \overline{\Psi(x)}$$

is equal to

$$(m(\lambda)\phi, \psi) - \frac{1}{\lambda + \mu} e^{-(\lambda+\mu)T} (m(\lambda)\phi, m(\mu)\psi).$$

We have denoted the residue of M at λ and μ by a lower-case letter. We have, moreover, again treated λ, μ, and T as real numbers. Since Λ^T is a projection that converges weakly to the identity as T approaches infinity, the first term in the difference is a hermitian form that dominates the second.

There is a final assertion to be verified, a simple form of the first.

Assertion C. *Suppose P is a proper maximal parabolic, then*

$$\sum_{\chi} \frac{1}{n(A)} \sum_{\Pi^P(M)} \int_{-\infty}^{\infty} \|I_P(\pi, \tau, \Delta)_\chi\|_1 \, d\pi$$

is finite.

Like Theorem 3.1 of [TF II], this assertion is deduced from the finiteness of

$$\sum_{\chi} \frac{1}{n(A)} \sum_{\Pi^P(M)} \int_{-\infty}^{\infty} \operatorname{tr}(I_P(\pi, \tau, f)_\chi) \, d\pi, \tag{1.8}$$

for a sufficiently smooth function f with compact support. This finiteness is verified by observing that the expression (1.8) is obtained by integrating the kernel of an integral operator on a compact manifold over the diagonal. Since the argument will be more than familiar to anyone with some acquaintance with the trace formula, I omit it.

Statement of lemma and its consequences

The principal assertion to be proved in this paper can be formulated as follows.

Proposition. *The sum of* $tr(\sigma(\Delta)_\chi)$ *over all classes* χ *attached to maximal proper parabolic subgroups and all representations* σ *occurring discretely in the space* $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ *is finite.*

Observe that the sum in question is a sum of positive terms, so that the proposition asserts that it converges absolutely. Since, as on p. 109 of [TF II],

$$\sigma(f) = \sigma(\Delta_1^{-n})\sigma(\Delta_1^n \star f),$$

we conclude that if f is at least $2n$ times continuously differentiable, then the assertion of the proposition remains valid for $\sigma(f)$ in place of $\sigma(\Delta)$. This is the claim of the Introduction.

The proposition will be deduced from the three assertions of the first section, supplemented by an elementary analytical lemma. Observe to begin that the operator $M_P^T(\pi,\tau)$ is, by its very definition, positive definite. To simplify the discussion, it will be convenient to take its restriction to the sum of the isotypical subspaces associated to a finite collection of irreducible representations of K and to a finite collection of π, but without any explicit indication in the notation. We may, however, whenever it is convenient, pass to the limit over an increasing sequence of collections that exhausts the set of all irreducible representations. The estimates we establish are uniform and therefore valid in the limit. The proofs, however, are less clumsy if carried out in a finite number of dimensions.

Choose orthonormal basis vectors $\Phi_j = \Phi_j(\pi,\tau)$ that are simultaneous eigenvectors for $I_P(\pi,\tau,\Delta)$ and $\Theta_\pi(\tau)$ with eigenvalues $\delta_j(\tau)$ and $e^{i\theta_j(\tau)}$ respectively. If $\Phi\star\Psi$ denotes the operator

$$\Upsilon \quad \to (\Upsilon,\Psi)\Phi,$$

then,

$$M_P^T(\pi,\tau) = -\sum_j \Big\{\theta_j^{'}(\tau)\Phi_j\star\Phi_j + \frac{1}{i}(\Phi_j^{'}\star\Phi_j + \Phi_j\star\Phi_j^{'}) + \frac{\sin(\theta_j(\tau))}{\tau}\Phi_j\star\Phi_j\Big\}. \qquad (2.1)$$

The eigenvectors and eigenvalues, although continuous functions of τ, may not be everywhere differentiable, but they are piecewise differentiable and that suffices. Formulas involving derivatives are understood to be valid where those derivatives exist.

Applying the operator $\Phi_j^{'}\star\Phi_j + \Phi_j\star\Phi_j^{'}$ to Φ_k and then taking the inner product of the result with Φ_k, we obviously obtain 0 for $k\neq j$, whereas for $k=j$ we obtain

$$(\Phi_j', \Phi_j) + (\Phi_j, \Phi_j'),$$

which is also equal to 0. Hence the diagonal terms of the matrix of (2.1) are

$$-\theta_j'(\tau) - \frac{\sin(\theta_j(\tau))}{\tau} \tag{2.2}$$

They are positive. To obtain those of the matrix that appears in (1.4) one multiplies by $\delta_j(\tau)$. Since the trace of any matrix is less than or equal to its trace-class norm, we conclude that the sum over χ, P, and j of

$$-\int \delta_j(\tau)(\theta_j'(\tau) + \frac{\sin(\theta_j(\tau))}{\tau})d\tau$$

is finite. Each term is of course positive.

We now state an elementary lemma that is central to our method; it will be proved in the following section. The number ϖ that appears is the area of the circle of radius 1.

Lemma. *Let $\theta(\tau)$ and $b(\tau)$ be two continuous functions on the interval $[\alpha, \beta]$, $\beta \geq \alpha \geq 0$, both functions being piecewise differentiable. Suppose that $b(\tau) \geq 0$ and that*

$$b(\tau) \geq \theta'(\tau) + \frac{\sin(\theta(\tau))}{\tau} \geq -b(\tau).$$

Then

$$\int_\alpha^\beta |\theta'(\tau)| d\tau \ \leq 6 \int_0^\beta b(\tau)d\tau + \varpi$$

and

$$\int_\alpha^\beta \left| \frac{\sin(\theta(\tau))}{\tau} \right| d\tau \leq 7 \int_0^\beta b(\tau) d\tau + \varpi.$$

The lemma immediately implies the analogous assertion for an interval $[-\alpha, -\beta]$. We deduce the next claim from the lemma.

Assertion D. *If P is a maximal proper parabolic subgroup then*

$$\sum_\chi \sum_{\Pi^P(M)} \int_{-\infty}^{\infty} tr\Big\{ -\frac{1}{i} \Theta_\pi^{-1}(\tau)_\chi \Theta_\pi'(\tau)_\chi \cdot I_P(\pi, \tau, \Delta)_\chi \Big\} d\tau$$

converges absolutely.

The absolute convergence of this expression is equivalent to that of

$$\sum_\chi \sum_{\Pi^P(M)} \int_{-\infty}^{\infty} \mathrm{tr} \Big\{ \frac{1}{\tau} \sin(\theta_\pi(\tau)_\chi) \cdot I_P(\pi, \tau, \Delta)_\chi \Big\} d\tau.$$

Because of Assertion C it is sufficient to treat the integral from -1 to 1. Our choice of Δ and standard facts about infinitesimal characters (see, for example, §8.6 of [RT]), imply that the numbers $\delta_j(\tau)$ are of the form $(\epsilon_j+\alpha\tau^2)^{-n}$, with α independent of π and $\epsilon_j \geq 1$. Hence, on the interval $[-1,1]$, estimates with $\delta_j(\tau)$ are equivalent to estimates with $(\epsilon_j)^{-n}$. This observation enables us to employ the lemma to infer that

$$\int_{-1}^{1} \left|\mathrm{tr}\left\{\frac{1}{\tau}\sin(\theta_\pi(\tau)_\chi)\cdot I_P(\pi,\tau,\Delta)_\chi\right\}\right| d\tau$$

is, apart from a factor that depends on α alone, less than or equal to the sum of

$$7\int_{-1}^{1} \left\|\left\{\frac{1}{i}\Theta_\pi^{-1}(\tau)_\chi\Theta_\pi'(\tau)_\chi+\frac{1}{\tau}\sin(\theta_\pi(\tau)_\chi)\right\}\cdot I_P(\pi,\tau,\Delta)_\chi\right\|_1 \, d\tau$$

and

$$\varpi\mathrm{tr}\{I^P(\pi,\tau,\Delta)_\chi\}.$$

That the sum of the first expression over $\Pi^P(M)$ and χ is finite is a consequence of the first part of Assertion A; for the second expression the sum over the same set is finite as a consequence of the second part of Assertion A applied to the group M.

The result is that the sum over χ, but of course only those χ associated to maximal proper parabolic subgroups, of either of the following two expressions is absolutely convergent. The outer summation in the first is over standard maximal proper parabolic subgroups.

$$\frac{1}{2\varpi}\sum_P\sum_{\Pi^P(M)}\frac{-1}{n(A)}\int_{-\infty}^{\infty}\mathrm{tr}\left\{\frac{1}{\tau}\sin(\theta_\pi(\tau)_\chi)\cdot I_P(\pi,\tau,\Delta)\right\}d\tau. \tag{2.3}$$

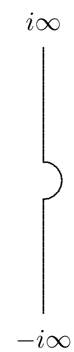
$$\sum_{\Pi(G)}\mathrm{tr}\{M_G^T(\sigma)_\chi\cdot\sigma(\Delta)_\chi\} \tag{2.4}$$

In the second expression, we have replaced π by σ in order to compare the two sums more conveniently. The factor $1/2\varpi$ has been introduced in the first for the same reasons. We evaluate the integrals by deforming the contour, observing that absolute convergence means, in particular, that the expression (2.3) continues to converge when the integrands are replaced by their absolute values.

Each of the integrals is of the form

$$\int_{-\infty}^{\infty} \operatorname{tr}\left\{ \frac{1}{i\tau}\Theta_\pi(\tau) \cdot I_P(\pi,\tau,\Delta) - \frac{1}{i\tau}\Theta_\pi(-\tau) \cdot I_P(\pi,-\tau,\Delta) \right\} d\tau.$$

They are evaluated by first deforming the contour to that of the diagram, on which both terms of the integrand converge, and then changing the variable for the second integrand, replacing τ by $-\tau$. The result is an integral over a contour with the bulge on the other side. It is deformed to the previous contour, leaving a residue of



$$\operatorname{tr}(M_\pi(0) \cdot I_P(\pi,\Delta)).$$

Applying the second part of Assertion A to the group M we see that the sum of this expression over $\Pi^P(M)$ and P is absolutely convergent, and, hence, of no further interest to us here.

The two integrals that remain now have the same integrand and are taken over the same contour. We may combine them into one, and move the contour over to the right. We take the final contour to be a vertical line a little to the the right of ρ_P. Assertions A and C imply readily that the sum over $\Pi^P(M)$ and χ of these integrals is absolutely convergent, so that they too are of no further interest. This leaves the residues. They arise only for parabolic subgroups conjugate to their own opposites.

If P is conjugate to its own opposite then $n(A) = 2$ and the combined integrand is equal to

$$\frac{-1}{2z}\Theta_\pi(z).$$

Thus each pole between 0 and ρ_P yields a residue

$$\operatorname{tr}\left\{ \frac{e^{-2\lambda T}}{2\lambda} m_\pi(\lambda) \cdot I_P(\pi_\lambda,\Delta) \right\} \tag{2.5}$$

The terms in (2.3) are labeled by π and by χ, but χ must be the class defined by the pair (π, M). The parameters σ and χ in (2.4) together determine a maximal parabolic P, a representation π in $\Pi^P(M)$, and a λ between 0 and ρ_P, such that χ is the class of (π, M) and the space $\mathcal{H}_G(\sigma)_\chi$ appearing on p. 926 of [TF I] is the set of residues of Eisenstein series attached to π and P at λ. We may add the term attached to σ and χ to that attached to π, P, λ, and χ to obtain a sum over λ, π, P, and χ. Once the sum over λ has been carried out, the remaining triple sum is absolutely convergent, but we cannot assert, without a closer examination of its terms, that the quadruple sum itself is absolutely convergent.

To compute

$$\mathrm{tr}\{M_G^T(\sigma)_\chi \cdot \sigma(\Delta)_\chi\} \tag{2.6}$$

for the σ associated to π and λ we use the formula of Assertion B, taking $\mu = \lambda$. Choose an orthonormal basis $\{\phi_j\}$ of $\mathcal{H}_P(\pi)_\chi$ consisting of simultaneous eigenfunctions for $m_\pi(\lambda)$ and $I_P(\pi, \Delta)$. Let the eigenvalue of $m_\pi(\lambda)$ for the eigenvector ϕ_j be a_j. Then $\{\frac{1}{\sqrt{a_j}}\phi_j | a_j \neq 0\}$ (more precisely, the residues at λ of the Eisenstein series associated to these vectors) can be taken as an orthonormal basis of $\mathcal{H}_G(\sigma)_\chi$.

The contribution of the terms

$$\frac{1}{a_j}(m_\pi(\lambda) \cdot I_P(\pi_\lambda, \Delta)\phi_j, \phi_j)$$

is $\mathrm{tr}\{\sigma(\Delta)_\chi\}$, a positive number. The contribution of the second part of the formula of Assertion B is for similar reasons given by

$$-\frac{e^{-2\lambda T}}{2\lambda}\mathrm{tr}\{m_\pi(\lambda) \cdot I_P(\pi, \lambda)\},$$

although now only one of the two factors $m_\pi(\lambda)$ has been removed by the denominators of the orthonormal basis. This cancels the contribution (2.5), and the conclusion is that

$$\sum_\chi \sum_{\Pi(G)} \mathrm{tr}\{\sigma(\Delta)_\chi\}$$

is finite, so that the proposition is proved.

Proof of lemma.

Since there are no longer any representations to deal with, we abandon in this section the symbol ϖ, replacing it by π. It is convenient to replace $\theta(\tau)$ by $\theta(\tau)+\pi$. The hypothesis then becomes

$$b(\tau) \geq \theta'(\tau) - \frac{\sin(\theta(\tau))}{\tau} \geq -b(\tau),$$

but the conclusion is unaltered.

The set of $\tau \in [\alpha, \beta]$ for which

$$\left|\frac{\sin(\theta(\tau))}{\tau}\right| > 2b(\tau)$$

is the union of a finite number of intervals open except perhaps at the points α and β. We number them from right to left as $B_1, \dots, B_r$. The complement is a finite number of closed subintervals $A_0, \dots, A_r$ that we also enumerate from right to left, taking A_0 to be empty if $\beta \in B_1$ and A_r to be empty if $\alpha \in B_r$.

On any of the intervals A_i we have

$$\left|\frac{\sin(\theta(\tau))}{\tau}\right| \leq 2b(\tau), \qquad |\theta'(\tau)| \leq 3b(\tau).$$

Hence

$$\sum_1^{r+1} \int_{A_i} |\theta'(\tau)|\, d\tau \leq 3 \int_0^1 b(\tau)\, d\tau. \tag{3.1}$$

In particular, if $r = 0$ the inequality of the lemma is certainly valid.

On any of the intervals B_i the function $\sin(\theta(\tau))/\tau$ is of constant sign. If it is positive we have

$$\theta'(\tau) \geq \frac{\sin(\theta(\tau))}{2\tau};$$

otherwise

$$\theta'(\tau) \leq \frac{\sin(\theta(\tau))}{2\tau}.$$

In either case, $\theta'(\tau)$ and $\sin(\theta(\tau))$ are of equal and constant sign on B_i.

If θ is any number let $\bar\theta$ be congruent to θ modulo $2\pi\mathbb{Z}$ and satisfy $-\pi \leq \bar\theta \leq \pi$. We shall show by induction on r that if $\varphi_0 = \theta(\beta)$ then

$$\sum_{i=1}^{r} \int_{B_i} |\theta'(\tau)|\, d\tau \leq |\bar\varphi_0| + \sum_{i=0}^{r-1} \int_{A_i} b(\tau)\, d\tau \tag{3.2}$$

The lemma follows immediately from (3.1) and (3.2). We may suppose that $\bar\varphi_0 = \varphi_0$.

Let the value of θ at the lower end of B_i be ψ_i and at the upper end be φ_i. Since $\sin(\theta(\tau))$ and $\theta'(\tau)$ are of constant sign on the interval, we have

$$\int_{B_i} |\theta'(\tau)|\, d\tau = |\varphi_i - \psi_i|.$$

We observe first of all that we may suppose that A_0 is empty or reduces to a point, because

$$\varphi_0 - \varphi_1 = \int_{A_0} \theta'(\tau)\, d\tau,$$

so that

$$|\bar{\varphi}_1| \le |\,\varphi_1| \le |\varphi_0| + \int_{A_0} |\theta'(\tau)|\, d\tau,$$

and the inequality (3.2) therefore a consequence of

$$\sum_{i=1}^{r} \int_{B_i} |\theta'(\tau)|\, d\tau \le |\bar{\varphi}_1| + \sum_{i=1}^{r-1} \int_{A_i} |\theta'(\tau)|\, d\tau. \tag{3.3}$$

Since $\sin(\theta(\tau))$ is of constant sign on B_i, we have $|\varphi_i - \psi_i| = |\bar{\varphi}_i - \bar{\psi}_i| = |\bar{\varphi}_i| - |\bar{\psi}_i|$ and $|\bar{\varphi}_i| \ge |\bar{\psi}_i|$. Consequently

$$|\varphi_i - \psi_i| \le |\bar{\varphi}_i|, \tag{3.4}$$

so that (3.3) is clear for $r = 1$. To complete the induction we need only show that for $r > 1$,

$$|\varphi_1 - \psi_1| \le |\bar{\varphi}_1| - |\bar{\varphi}_2| + \int_{A_1} |\theta'(\tau)|\, d\tau. \tag{3.5}$$

There are three possibilities to consider. Suppose first of all that $\theta(\tau)$ assumes the value $2n\pi$ on A_1 with $n \in \mathbb{Z}$. Then

$$|\bar{\varphi}_2| \le |\varphi_2 - 2n\pi| \le \int_{A_1} |\theta'(\tau)|\, d\tau,$$

and (3.5) follows from (3.4). To deal with the remaining cases, we use the relation

$$|\varphi_1 - \psi_1| = |\bar{\varphi}_1| - |\bar{\psi}_1| = (|\bar{\varphi}_1| - |\bar{\varphi}_2|) + (|\bar{\varphi}_2 - \bar{\psi}_1|).$$

If $\bar{\varphi}_2$ and $\bar{\psi}_1$ have the same sign, then

$$|\,|\bar{\varphi}_2| - |\bar{\psi}_1|\,| = |\bar{\varphi}_2 - \bar{\psi}_1| \le |\varphi_2 - \psi_1| \le \int_{A_1} |\theta'(\tau)|\, d\tau,$$

so that (3.5) is valid. Suppose they have opposite signs and $\theta(\tau)$ does not assume a value in $2\pi\mathbb{Z}$ on A_1. If $|\psi_1| \geq |\varphi_2|$ then $|\varphi_1 - \psi_1| \leq |\bar\varphi_1| - |\bar\varphi_2|$, and (3.5) is immediate. If this inequality does not obtain then in passing from the value ψ_1 to the value φ_2 on A_1 the function θ must take on a value ψ such that $\bar\psi = -\bar\varphi_2$ and $\text{sign}(\bar\psi) = \text{sign}(\bar\psi_1)$. Hence,

$$\big|\,|\bar\varphi_2| - |\bar\psi_1|\,\big| = \big|\,|\bar\psi| - |\bar\psi_1|\,\big| = |\psi - \psi_1| \leq \int_{A_1} |\theta'(\tau)|\,d\tau.$$

Appendix

For orientation I review the procedure of Arthur's papers to see what value one obtains for the integer n appearing in (1.4), although this is unnecessary for our present purposes. It is easier to start from the final step in §3 of [TF II]. There is an integer r_2, that has yet to be defined, and n is so chosen that the equation

$$\Delta_1^n g = \delta + h,$$

implies that g has continuous derivatives up to order r_2 if the function, or perhaps better distribution, h is infinitely differentiable and δ is a delta-function at some point. Applying the elliptic regularity theorem and the Sobolev embedding theorems (Th. 20.1 and Cor. 6.1 of [PDE]), we obtain $2n - d \geq r_2 + \frac{d}{2}$ or, as a possibility,

$$n = \left[\frac{r_2}{2}\right] + \left[\frac{3d}{4}\right],$$

if we regard δ as lying in the space W^{-d}, d being the dimension of G. Square brackets denote the smallest integer greater than or equal to a given real number. The same considerations imply that the integer r_0 introduced in §4 of [TF I] may be taken to be

$$r_0 = 2\left[\frac{3d}{4}\right] + 1.$$

The δ-function at the identity on K is a sum of the characters,

$$\delta = \sum_\sigma d_\sigma \chi_\sigma.$$

If this were an expansion valid in the space of measures on K, the K-finite functions in any one of the usual function spaces on$G$ would be dense in that space. It is not, but if Δ_2 is the analogue for K of Δ_1 and if l is such that

$$2l > \dim(K)$$

then

$$\sum_\sigma \Delta_2^{-l} d_\sigma \chi_\sigma$$

converges in the mean-square and therefore in the space of measures. Then $l_0 = 2l$ is the integer introduced in §4 of [TF I] that appears in the definition of r_2 in §3 of [TF II] and can be taken to be $\dim(K) + 1$.

$$r_2 = r_0 + l_0 + \sum_i \deg(Y_i)$$

An inspection of the proof of Theorem 3.2 of [TF II] makes it clear that to take the sum of the degrees of the Y_i is exaggerated; the maximum suffices and it is the rank $\mathrm{rk}(G)$ of G. The conclusion is that we may take

$$n = 2 + 2[\frac{3d}{4}] + [(\dim(K) + \mathrm{rk}(G))/2].$$

This should be compared with Cor. 3.15 of [TC].

References

[TF I] James Arthur, *A trace formula for reductive groups I: Terms associated to classes in G(**Q**),* Duke Math. Jour., v. 45, 911-952, (1978)

[TF II] —, *A trace formula for reductive groups II: Applications of a truncation operator,* Comp. Math., v. 40, 87-121 (1980)

[PW] —, *A Paley-Wiener theorem for reductive groups,* Acta Math., v. 150, 1-89 (1983)

[IP] —, *On the inner product of truncated Eisenstein series,* Duke Math. Jour., v.49, 35-70 (1982)

[FD I] —, *On a family of distributions obtained from Eisenstein series I: Application of the Paley-Wiener theorem,* Amer. Jour. Math., v. 104, 1243-1288 (1982)

[FD II] —, *On a family of distributions obtained from Eisenstein series II: Explicit formulas,* Amer. Jour. Math., v. 104, 1289-1336 (1982)

[EE] H. Donnelly, *Eigenvalue estimates for certain noncompact manifolds,* Mich. Math. Jour., v. 31, 349-357 (1989)

[RT] A. W. Knapp, *Representation theory of semisimple groups,* Princeton University Press (1986)

[ES] R. P. Langlands, *On the functional equations satisfied by Eisenstein series,* SLN in Math., 544 (1976)

[TC] W. Müller, *The trace class conjecture in the theory of automorphic forms,* Ann. of Math, v. 30, 473-529 (1989)

[PDE] J. Wloka, *Partial Differential Equations,* Cambridge University Press (1987)