SUPPLEMENT

The first draft of the review was sent to a number of friends and colleagues, not all of whom had the time to respond, but some did and I am very grateful to those friends, especially James Arthur, Dipendra Prasad, Claude Levesque and Freydoon Shahidi, who drew my attention to misprints, blunders and stylistic lapses, to James Milne, who corrected both solecisms and an unintended historical misrepresentation, to Bill Casselman and to Anthony Knapp, who carefully explained to me the many difficulties a reader for whom the field was largely new would have with my presentation. At the same time, style and clarity aside, there are a number of aspects of the review about which I was more than a little uneasy.

In attempting to describe my impressions of the major contours of the field as a whole, I introduced a square of objects (automorphic representations, motives, Hecke algebras, and Galois representations) as a key element. I was relieved and delighted when all of the three specialists who replied, Michael Harris, Richard Taylor and Jacques Tilouine, responded positively to it. That they had reservations is hardly surprising. There was no reason that my observations be right on the mark. Their suggestions to me will be helpful to the reader as well, but rather than try to incorporate them in a revised and much longer, much more difficult review, I prefer to present the comments as they came to me, in informal e-mail messages or brief pdf files. There is no particular value in presenting myself as an expert when I am not. It will take me a great deal of time to digest all that they wrote, and the reader, even the novice, is better off with information coming straight from the horse's mouth.

The suggestion that ultimately more use might be made of functoriality, especially for tori, reflects, of course, my own experience and my own contributions to the theory of automorphic forms. So I was pleased to discover that it was a point of view already well represented among specialists.

A final point that troubled me and about which I am still uneasy is the presentation of the Selmer group as a Galois-theoretic p-adic form of the extensions that appear in the theory of mixed motives. I have not seen this parallel clearly mentioned in the literature, although I suppose it was familiar to specialists. This is confirmed by the lack of comment on it. The reader is warned, however, that I have no positive evidence that it is cogent and generally accepted. As a whole my comments on material related to the definition and properties of p-adic L-functions especially to the Selmer group and to the main conjecture leave a great deal to be desired in the way of precision and suggestiveness. So I am extremely grateful to Ralph Greenberg for communicating a number of essential insights into the difficulties to be overcome and several suggestions for further reading.

The comments of Harris and Taylor correct my review on a number of points connected with $\mathfrak{H}_p$. My response – and it will I hope be the response of the reader as well – is simply that I will have to study their remarks and references until I have understood them. They confirm the sentiment expressed in the first paragraph that modern number theory is experiencing an almost unprecedented eclosion. There are indeed a number of fascinating suggestions that bear a lot of reflection scattered among their informal remarks.

It will also be clear to all readers of the review that, for lack of competence, I steered clear of many technical difficulties. I did not come to grips with Hida's technical achievements because I was not and am not yet in a position to appreciate them. So, although my uneasiness about Hida's style appears to be shared, the emphasis of Harris and Tilouine on the importance of Hida's aims and accomplishments is a welcome corrective to any tepidity that may be present in my review. It was a result of ignorance. On seeing the draft of the review, more than one mathematician expressed his indebtedness to the personal influence of Hida.

The remarks of Harris, Taylor and Tilouine are presented below in the order and the form in which I received them. After that, come observations of Ralph Greenberg and some questions of Laurent Clozel and, finally, the first draft of the review, not much different from the final version. It is, however, the version to which the comments were directed.

Some apology for not following the advice of Tony Knapp is necessary. Casselman had commented that '`your review is a tough read.`' Knapp was blunter, explaining,

> `I have tried to put myself in the position of a reader who might want to know a little more about the general field in question, say a number theorist who knows little about representations or a representation theorist who knows little about number theory. I made some notes as I read the first two and a fraction pages and then stepped back to guess at the reader's reaction. My guess at that reaction is 'Do I have to understand all this in order to understand any of the book?'`
> `That reader might well put the review aside at that point, be angry with you, and go away having learned nothing about the book or the field. In my own case I persevered, though skipping things here and there. When I got to page 10, I saw what your answer might be: 'Maybe yes. But maybe it is worth it' And I realized that the first paragraph of the review was continuing that sentiment by adding, 'That being so, let me supply some background' My key point to you is that it would be much better if the reader had seen this assessment of yours (the one on page 10) in the first page or two. Then the reader would be grateful that you gave a frank opinion and would be better positioned to decide how much further to read in the review.`

So, even though I managed to meet experts more than half-way, I was running the danger of alienating not just one reader but a broad class of readers, those who wanted to know what the book was about but who, unlike Tony Knapp, would be unwilling to persevere. At the same time, the editor, Robert Devaney, was satisfied, '`I think anyone who reads this review will very much enjoy what you have written (both from a mathematical and personal standpoint).`' He was indeed enthusiastic, '`Yes, the review is long, but to be honest, I never realized this as I was reading the review. It seems perfect as is for the Bulletin.`' So I was, and am, torn. There were classes of readers who found the review more than satisfactory; yet at the same time there was a danger of alienating many others. The absence of a response from a number of specialists to whom I sent the review may be a further sign of this.

The arguments in favor of a serious rewriting are strong. My initial hope, when undertaking the review, was that of a mathematician whose principal competence lay at the margins of the field and who wished no more than a modest understanding of the meaning of the many conjectures about the values of L-functions and their relations to algebraic geometry. As the review developed, my stance changed and my inclination now would be to attempt, so far as time and talent allowed, to acquire some mastery of the field and to compose a text that revealed both its fine lines and its broad contours. Casselman's observation that '`my impression is that you have in mind a much longer version with drastic cuts. That's fine, editing is undoubtedly necessary, but I also have the impression that you have edited this review for the pleasure of experts, and that therefore the cutting room floor is filled with the sort of stuff The Naive Reader would appreciate.`' has therefore an element of truth. I had in mind explaining more, but the editing was not a matter of choice but of necessity. I did not understand enough to say more. The limitations of space imposed by the review were my salvation. Nevertheless, even with more more knowledge at my disposal, I would still have to decide whom to address. It is not clear. For the moment, since I am unable to satisfy fully any group, there is no need to do so. Having some evidence that I have not offended the specialists and that the text, if read without great expectations, has something to offer the uninitiated, I decided to leave well enough alone, to correct all misprints and all misleading references and to mitigate all doubtful affirmations with the corrections of the experts, but to let the original text otherwise stand. There is a great need for a longer text that also devotes adequate space to the very many topics given, in spite of their importance for the book at hand, short shrift in the review: mixed motives; the Bloch-Kato conjecture; the main conjecture. We can all dream of writing it.

A text, even a book, that explained – with as few prerequisites and as few technicalities as possible and with no reference to black boxes such as étale cohomology – the relation between algebraic irrationalities and diophantine equations or even between algebraic irrationalities and the geometry and topology of algebraic varieties would also be very welcome, but even more difficult to compose.

On the other hand, Tony Knapp was addressing difficulties and suggesting improvements related to neither extreme, but just to the problem of making the text more widely accessible without major modifications either to content or to goals. Nevertheless even though I considered it carefully, I ultimately ignored important aspects of his very sound advice. There are reasons. It was only slowly, in the course of writing the review, that I appreciated that p-adic L-functions and the attendant problems were only one corner of a relation that expressed the strategy developed over the last two decades for dealing with a broader, and at the same time much more central, matter, that of the identity of the class of motivic L-functions with a class of automorphic L-functions. This is not an issue in the book although the importance of the book is not independent of its relevance to it. The author is certainly not unaware of this.

I do not admit frankly in the review that my principal concern is with the larger question, but this may be transparent. It is not until we are halfway into the review and have some familiarity with the four elements of the nexus and the difficulties associated to them that we arrive at the book itself. So I have cheated: I have introduced a topic that strongly affects the size of the potential audience for this and related books as well as my judgement of their importance but is none the less a topic not strictly belonging to the book's subject matter. I have moreover deliberately kept the book in the background until I had completed the introduction of the broader issue. I have no regrets, but this does entail an obscurity that Knapp was trying to remedy. Although I am not at all certain just what the ultimate role of p-adic L-functions and Shimura varieties in the developing arithmetic will be, they alone do not capture its grandeur!

Comments of Michael Harris

Dear Langlands,

When you wrote me last summer I somehow didn't guess that the book you had agreed to review was Hida's latest book. This is an essential reference for my project with Li and Skinner and although I have immense admiration for Hida for his personal qualities as well as for his mathematical vision, I have some sympathy for the comments on p. 10 of your review.

There is one troubling point in Hida's book I have not been able to settle. One of his main theorems is the irreducibility theorem for Igusa towers, proved in section 8.4 and crucial for applications to p-adic L-functions. This connection was discovered by Katz and Ribet's proof of irreducibility for Hilbert modular varieties was one of the main steps in the Deligne-Ribet construction of p-adic L-functions for totally real fields. Chai has a simpler proof for PEL Shimura varieties with point boundary components (among other cases) and is probably the best-informed specialist on the question, but he has told me he is unable to understand one of the main steps in Hida's proof, namely the appeal to Serre-Tate coordinates at the end of the second paragraph of p. 371, which seems implicitly to depend on the conclusion of the theorem being proved. I have not tried to sort this out. You may want to check with Chai on the status of this argument.

The specialists in p-adic automorphic forms have been slowly groping toward an appreciation of the role of functoriality, especially during the ongoing special semester at Harvard on eigenvarieties. Beyond its original motivation as a review of Hida's book, your article will be a fundamental contribution to understanding this rapidly growing field in the context of functoriality, and it comes at just the right time.

I have a few comments about the content of the review. The first is to place Hida's choice to work with holomorphic forms (de Rham cohomology) rather than Betti cohomology in historical perspective. In fact, Hida had developed his theory for Betti cohomology in considerable generality, and although he worked it out in detail for groups G which at p are isomorphic to GL(n) (references [H95] and [H98]), the method was sufficiently general to apply to symplectic groups (the article of Tilouine and Urban) and indeed to general quasi-split groups (the thesis of Tilouine's student David Mauger -- which I think only treats groups split at p). However, proof of the vertical control theorem, which is perhaps the main result of Hida theory, runs into difficulty for locally symmetric spaces of dimension > 1 because of the possible presence of torsion in cohomology. I will return to the issue of torsion below. Hida was very pleased in the late 90s when he realized that he could work with holomorphic forms in higher dimension. Torsion is less of a problem, because only coherent cohomology in degree 0 is involved, but there are new issues arising from arithmetic compactification.

Probably the most significant omission from your review is any reference to the growing literature on non-ordinary deformations (sometimes called "positive slope" deformations by analogy with the case of classical modular forms). This subject was initiated by Coleman, at Mazur's suggestion, and has grown into the field of eigenvarieties which has been occupying the time and energy of a great many people at Harvard all spring. Eigenvarieties are a more geometric approach to what you call the $\mathfrak{H}_p$ than one finds in Hida's work; in particular, they really live over a characteristic zero base rather than over rings of mixed characteristic. I'm not a specialist, but I can point to two events that indicate that the field has matured: Kisin's work relating the local structure of eigenvarieties to local Galois deformation rings, in his work on the Fontaine-Mazur conjecture; and the article of Skinner-Urban using p-adic (non-ordinary) deformations from non-tempered cohomology classes of Siegel modular threefolds to tempered classes in order to construct infinite subgroups of Selmer groups of elliptic modular forms. Hida's theory corresponds to the part of the eigenvariety with the best analytic behavior. My impression from talking to Mazur is that he and the other specialists are now speculating about functoriality in terms of general eigenvarieties but only expect to be able to confirm their speculations for the (nearly) ordinary locus constructed by means of Hida theory.

I'm not sure I understand your comment on p. 10 regarding the definition of $\mathfrak{H}_p$. As I indicated above, Hida has a very general definition for groups isomorphic to GL(n) at p -- including many classes of unitary groups. The case where G(R) is compact modulo center has been studied quite generally, in part because it is clear they are well adapted to the Taylor-Wiles method. Deformations with fixed Hodge-Tate parameter, as in Wiles' theory, are studied for definite unitary groups in my unpublished article with Taylor on the Taylor-Wiles method for unitary groups, written between 1996 and 1998 and mostly incorporated into the article with Clozel that you cite. The extension to (nearly) ordinary deformations in Hida's sense is routine and well understood by experts. The general (nonordinary) version of this theory was developed my the thesis of my student Chenevier, completed in 2003. In this way Chenevier was the first to construct eigenvarieties -- he used "variété de Hecke" as the French translation -- of dimension greater than one.

In his thesis Chenevier also used constructed the corresponding families of n-dimensional Galois representations (in general he has to work in the setting of pseudorepresentations, but in most cases it gives what you would expect). He has given a course on these results and their arithmetic applications at the Harvard eigenvarieties semester. This answers your implicit question in the second paragraph of p. 12 regarding the theory of parametrized families of Galois representations of dimension greater than two. The Galois deformation theory corresponding to automorphic forms on unitary groups of fixed weight is, of course, the subject of my unpublished article with Taylor, its successor with Clozel, and Taylor's most recent article on the subject. Together with Kisin's work, this latest article of Taylor eliminates all but one remaining technical obstacle to applying Wiles' approach to all Galois representations arising from Shimura varieties. The remaining technical obstacle has to do with weights that are large relative to the congruence prime; Kisin thinks this will be resolved in the setting of the "p-adic Langlands program." (The Taylor-Wiles method for GSp(4) is the subject of a article of Genestier-Tilouine; the methods are roughly the same as those in our work on unitary groups but the geometric problems are technically more difficult.)

More information about Chenevier's course is at

http://www.math.harvard.edu/ev/newsletter.html

(the item dated March 1). You may also be interested in some of the other items on this page.

With regard to your comment on p. 4 on "passages from one [p-adic] leaf to another," Taylor's method, which is a geometric version of weak approximation, is surprisingly powerful. My paper with Shepherd-Barron and Taylor extends the method to (certain kinds of) representations of any dimension. Apart from the problem of finding local points on moduli spaces with prescribed ramification at the congruence primes, which I don't think is insurmountable, there seems to be no obstacle in principle to applying the method to prove that any compatible system of l-adic representations of the Galois group of a CM field F with the right polarization (an archimedean sign condition generalizing the odd condition for GL(2)) can be connected by a series of p-adic deformations to a compatible system coming from Shimura varieties, up to replacing F by a CM extension. If the Hodge-Tate numbers are right then by a series of deformations to monomial representations we should even deduce that the L-function of the original compatible system has a meromorphic continuation satisfying the expected functional equation. This is still a long way from the general problem but it may be more than an isolated class of cases. In any event, Taylor's potential version of Serre's conjecture is the starting point of the Khare-Wintenberger proof of the original conjecture.

I wanted to say something about torsion in the integral cohomology of locally symmetric spaces. Already while he was writing his thesis, Taylor was convinced that if G is GL(2) of an imaginary quadratic field K, then the torsion in the cohomology of the corresponding locally symmetric space has associated 2-dimensional Galois representations (of the absolute Galois group of K). After you wrote me last August I learned from Mazur that he and Calegari have been attempting to formulate precise conjectures about the p-adic representations obtained by piecing together those (conjecturally) attached to torsion classes for different coefficient systems. Their conjectures inevitably lead them to questions about functoriality. A typical question, put very roughly, is to determine the locus of these l-adic representations in the p-adic eigenvariety associated to Siegel modular forms of genus 2. The idea is that induction of the 2-dimensional Galois representation of K to a 4-dimensional Galois representation of Q carries the right kind of polarization (the archimedean sign condition again) to be in the eigenvariety for holomorphic Siegel modular forms, or rather in its Galois version, the one you denote $\mathfrak{G}_p$. On the other hand, the base change to K of a p-adic family of elliptic modular forms, twisted by appropriate (variable) motivic Hecke characters, gives rise to another locus in the Siegel modular eigenvariety; here one can make sense both of $\mathfrak{G}_p$ and $\mathfrak{H}_p$. Calegari and Mazur believe, first of all, that one can identify $\mathfrak{G}_p$ and $\mathfrak{H}_p$; next, that these two loci described above meet transversally, or at least that they have no common components; finally, that the union of the two loci corresponds to the polar locus of an appropriate p-adic L-function (I think it would be defined by the Tannakian condition corresponding to the map of L-groups). They have a fair amount of numerical evidence for this, as do Ash and his collaborators for the related case of cohomology of the locally symmetric space attached to GL(3) over Q (here there is a pro-torsion locus and a "classical" locus obtained as the image of the symmetric square from GL(2) of elliptic modular forms).

In this sense, the number theorists have already been "reflecting" on the "passage to the primed objects" as you put it in the next-to-last paragraph of p. 12. The surprising answer seems to be that functoriality for p-adic automorphic forms may not simply be obtained by p-adic completion of functoriality for complex automorphic forms; one has to allow pro-torsion cohomology classes as well. Some of us have ideas for actually proving some cases of these conjectures that are not totally far-fetched, but it's far too early to say anything definite.

I apologize for the long-winded comments on your review. Please feel free to quote anything you find useful; no attribution is necessary.

Best regards,

Michael Harris

As Harris suggested, I asked Chai about Hida's proof of the irreducibility theorem for Igusa towers. His reply follows.

Response of Ching-Li Chai

Dear Prof. Langlands,

It is a surprise to get a message from you, I have to say. I am in Amsterdam for a conference but would like to get back to you with some preliminary response first. I will write to you after going back to Philadelphia on Thursday (June 1st).

Unfortunately I still have not gone through Hida's book carefully enough to draw any conclusion that I can " state to the world". The only thing I can say is that I don't understand that proof. I remember that the proof uses embdedding into a Siegel modular variety, and Hida wanted to draw information from the valuation defined by the q-expansion for Siegl modular varieties.

One of the reasons I did not pursue it further is that there are other proofs of that statement. I know of at least two proofs using ideas different from Hida's. One is a comination of Ribet's original method (for Hilbert modular varieties), then using Hecke correspondence which fixes a "hypersymmetric point" of the moduli space in characteristic p. (The hypersymmetric points correspond to abelian varieties whose endomorphism ring is "as large as possible" under a given slope constraint. For ordinary abelian varieties, the hypersymmetric ones are those which are isogenous to a product of copies of a fixed elliptic curve defined over $\bar{\mathbb{F}}_p$.) On the other hand, Ribet's orginal method also works. (There seems to be a prevailing opinion that Ribet's method works only when the target of the p-adic monodromy representation is a commutative. This is not the case. Incidentally, I gave a sketch of a proof of this just two hours ago.)

My experience has been that Hida usually has good ideas. So it is entirely possible that the idea sketched in the book works, and I just failed to understand it. There was a result of Hida that I did not undersand when I heard it in a talk at Toronto. But eventually the proof worked.

Best regards, Ching-Li

RPL.There is an important supplement to these remarks, again from Chai.

I hope there is still time to update/revise my previous message about Hida's method on p-adic monodromy for subvarieties of modular varieties of PEL-type in characteristic p, defined by p-adic invariants.

Several email messages from Hida helped me to finally understand the key idea in his proof: the local stabilizer subgroup in Hecke correspondences at a point can be effective exploited to produce elements in the image of the p-adic monodromy. The lifting to characteristic zero, q-expansion, and embedding to Siegel modular varieties in Hida's argument can be replaced by an irreducibility statement which resulted from group theory and ℓ-adic monodromy. (Pure and Applied Math. Quarterly,1, 2005, special issue in memory of Armond Borel, 291-303, Prop. 4.4 and 4.5.4.)

When applied to a hypersymmetric point, the above argument proves the maximality/surjectivity of the p-adic monodromy map by "pure thought". Moreover it applies to "leaves", subvarieties defined by an isomorphism type of p-divisible group with prescribed endomorphisms and polarization.

The above statements are of course subject to error and need to be checked carefully. But any error and imprecision would be my responsibility. The point is that Hida had a good idea, and it works.

Comments of Richard Taylor

Dear Robert (if I may),

I enjoyed your review of Hida's book. Here, for whatever they are worth, are a few comments that came to my mind.

1) Concerning your diagram on page 5: I do not think the vertical dashed lines should be thought of as ''correspondences''. For a general reductive group G the spaces h_p will be much bigger than a and the spaces g_p will be much bigger than m. (Forgive my inability to reproduce gothic letters in this message.) Most clearly I presume g_p is the space of p-adic Galois representations of the absolute Galois group of F into the L group of G - in some suitable sense. Within this space only those respresentations which are unramified almost everywhere and de Rham (in Fontaine's sense) above p will correspond to an element of m. For GL_2/Q these so called 'geometric' points may be distributed rather regularly, but in general they seem to be distributed rather randomly. For example if one looks at GL_2 over an imaginary quadratic field K and looks at Galois representations which become diagonalisable on restriction to decomposition groups above p one sees whole componets with no 'geometric' points, positive dimensional components with one 'geometric' point and postive dimensional components with a dense set of 'geometric' points. We believe that whenever we get a geometric point the p-adic representation is part of a compatible family and so we get corresponding points on all the other leaves g_l. Moreover these 'geometric' poinst should be in bijection with m. However it seems that there can be whole components of g_p with no geometric point and hence no relation to m nor to the other g_l. This seems to happen both for GL_2 over an imaginary quadratic field and for GL_3 over Q.

If one believes in your square, and I am inclined to, one should believe that something similar happens on the left hand side. As you say there is a problem in defining h_p. But it seems to me (and I think others) that it should be some sort of space of p-adic automorphic representations. There should be some correspondence between h_p and g_p - close to a bijection. Hence the different h_p should only touch at certain special points, which for general G seem to be rather randomly distributed. I would like to think of the usual automorphic forms as h_infinity, and the points where h_infinity meets one (and hence presumably all the) h_p would just be the set of arithmetic automorphic forms you call a.

For groups like GL_2 over an imaginary quadratic field and GL_3 over Q one can construct pieces of h_p which are far away from anything in a using torsion in the cohomology of quotients of the symmetric space.

In summary m provides the glue between the different sheets g_p. Similarly a provides the glue between the different sheets h_p and also h_infinity. The sheets h_p and g_p should correspond, matching a with m. Unfortunately I have no idea what g_infinity should be - if indeed it exists!

Note that this is exactly the picture one has for GL_1 over a field K. h_p would be the space of continuous p-adic characters of the idele clas group. h_infinity the space of grossencharacters of the idele class group. g_p the space of continuous p-adic characters of the absolute Galois group of K. m would be the space of those p-adic characters which are 'geometric' and hence fit into compatible systems. (See Serre's book on abelian l-adic representations.) a would be the set of grossencharacters of typoe A_0, ie arithmetic in your sense. Everything fits together, but you already see that a can be quite sparse in h_p and h_infinity. Take for instance K to be a cubic field with two infinite places. (However the picture for GL_2 over an imaginary quadratic field seems to be even less regular than this.)

The 'geometric' points seem to be fairly uniformly distributed in g_p, i.e. a seems to be fairly uniformly distributed in h_p, exactly in the case that G(R) has an inner form which is compact mod centre.

You are right that the thrust of Wiles, Taylor-Wiles and further developments along these lines, is that if a point of h_p matches a point of g_p then locally h_p looks just like g_p. However these arguments only seem to have a hope of working in the case that G(R) has a compact mod centre inner form, i.e. that the points of a are dense in h_p.

You are also right that these arguments become much more powerful when combine them with switching from h_p to h_l to h_l' ... and eventually back to h_p (at points in a). This allows you to travel much further in h_p as points which are far apart p-adically may be close l-adically. One may even move from one 'component' of h_p to another. Primitive arguments of this form are found in Wiles' Fermat paper (and indeed predated it), but they have been developed greatly, most recently by Khare and Wintenberger.

(For GL_2 over Q Matt Emerton at Northwestern has been developing a very nice picture, which you might find interesting. I am not sure how much he has written up, but you could check his website.)

2) Concerning the second paragraph on page 10, beginning 'At present, to give any definition': Firstly I don't think this is true. In my (unpublished) thesis I looked at GL_2 over an imaginary quadratic field. I think Hida subsequently looked at GL_2 over any number field. There is ongoing work of Ash and Stevens on GL_n over Q for n>2. However what we prove is weaker than in the case that there is a Shimura variety or inner form G(R) compact modulo centre, because h_p can be smaller than one might guess at first. (In technical parlance 'the eigenvariety will not be flat over weight space'.)

Concerning your comment about using groups which are compact at infinity, I believe that Chenevier has worked out the theory of the eigencurve for unitary groups of any rank which are compact at infinity. Similarly Clozel, Harris and I make heavy use of these definite unitray groups in our recent preprint.

3) Concerning the second paragraph on page 11 which begins 'If l is not equal to p ... It may as well be fixed': I disagree. These deformations are as you point out much more limited, but their study is the basis of both Wiles' paper on Fermat's last theorem and of Taylor-Wiles. They are definid over much smaller rings (eg Z_p[[T]]/((1+T)^p-1) as opposed to Z_p[[T]]). However their study has led (so far) to far more interesting theorems than the study of deformations of the restriction to decomposition groups above p.

Also I do not think the odd/even dichotomy is the same as the ordinary/non-ordinary dichotomy. For higher rank groups with G(R) having a compact mod centre inner form there is still an odd even dichotomy. For instance if the L group is GSp_2n one asks that the multiplier of complex conjugation is -1. I know a similar criterion for unitray groups. The ordinary/non-ordinary dichotomy is different. There are ordinary even representations, but they don't move in big families. For GL_2 over Q they don't move at all. Suppose now we are in the odd case, then in both the ordinary/nonordinary cases one expects families of automorphic forms of the same dimension - the 'eigencurve' or 'eigenvariety'. (Technically I only know this in the 'trianguline' case, which is much weaker than ordinary but not completely general - I am not sure what is expected in teh non-trianguline case.) In the non-ordinary case it is hard to give these families an integral structure. The specialisations in weights k and k' congruent modulo p^c will only be congruent modulo some smaller power of p. What distinguishes the ordinary case, and makes it technically much easier to work with, is that there is a very nice integral structure. Hida, who first discovered these families, seems only ever to work in the ordinary case. But Coleman, Mazur and others have generalised his theory.

4) Concerning page 13, paragraph 2 beginning ``The functions are to be p-adic ...'': Maybe they should be defined on the whole of h_p or g_p, just as non-arithmetic automorphic forms have L-functions. They would need however a direct definition. They could only be related to complex L-functions at points in a or m. I don't know much about it, but I believe that for GL_2 over Q there are hints in this direction in work of Kato, Colmez and Emerton.

I hope these comments might be some help to you. Best wishes,

Richard

The comments of Tilouine came as a pdf-file!

Comments on a recension of Hida's book

J. Tilouine

May 28,2006

I read with great interest your recension. I kind of forgot my own review of Hida's book on p-adic automorphic forms, so I had to reread it to compare with your comments. First, I subscribe completely to your presentation of the modern approach of the arithmetic of automorphic forms via the four elements of the nexus drawn on page 5. Apart from one or two misprints, I don't see anything to modify up to page 9.

Then, concerening comments on the book on p. 10 and p. 12 of the text you could maybe develop the remark of l. 10 from bottom p. 10 about the beautiful program hinted at in the introduction of Hida's book; in particular his vision of an integral version of Shimura's reciprocity law relating Galois theory of Igusa towers inside theKottwitz p-adic integral model of Shimura varieties to (non-abelian) Iwasawa theory of certain number fields.

The fact that the book is (statistically) not for graduate students is sad but true. That it should have been edited more carefully is also obvious. But I still believe it is edifying (hence challenging).

I think that the mere existence of your review speaks in favor of the book, but an emphasis on the vision it contains/suggests could be made slightly more strongly. With all its defects (and inessential errors) I think one should say it is really important.

Comments of Ralph Greenberg

Dear Professor Langlands,

I looked through the article that you sent me and have a few comments. I'm sorry to have taken so long. I hope my comments will still be helpful.

Concerning the discussion of the main conjecture on page 16. One of the difficulties of the topic is that the characteristic function of a Selmer group is only well-defined up to a unit in R. Only the ideal (called the characteristic ideal) is well-defined. Thus, the main conjecture asserts that the characteristic ideal has a generator which is , essentially, the p-adic L-function. Even the p-adic L-function is not very well-defined. The Kubota-Leopoldt p-adic L-functions are precisely defined in terms of the complex values of Dirichlet L-functions. The Mazur-Swinnerton-Dyer p-adic L-function for an elliptic curve E/Q is also well-defined because the period can be chosen canonically. (Note: This function was defined at first only when E has good, ordinary reduction at p.) This is a function of one p-adic variable and corresponds to twisting the Tate module for E by characters of the Galois group Gamma that you define. But the analogous p-adic L-function for a modular form of arbitrary weight is not well-defined. One has to choose a period. The situation becomes more complicated when one considers p-adic L-functions of more variables than just the cyclotomic variable. For example, one might have a "Hida family" of ordinary modular forms of varying weight k. Then the interpolation property becomes less clear. In each weight, it will coincide with the one variable p-adic L-function for that modular form with a suitable complex period, but only up to multiplication by some mysterious p-adic unit. Thus, I'm not sure if it is accurate to say that the main conjecture can be formulated directly in terms of the complex L-function and the p-adic representation in general.

Also, concerning the proofs of the main conjecture, the methods so far are based on establishing a divisibility in one direction or the other and then somehow showing that the corresponding quotient must be a unit in R.

Remarks about page 17. I can say a little about why I chose a discrete Galois module, namely what you call V^ , for the purpose of defining a Selmer group. The problem with looking at extensions of V (as in (7)) is that H1(Gal(Qbar/Q) , V) doesn't give very much information. This can be illustrated by the Tate module T_p(E) for an elliptic curve E. Assume, for simplicity, that E(Q) has no p-torsion, which will usually be the case. Then H1(Gal(Qbar/Q), T_p(E)) is a free Z_p-module of some rank. One could define a Selmer group, but, by itself, it would simply be a free Z_p-module and its rank would be equal to rank(E(Q)). The Tate-Shafarevich group would not be present. One would have to consider H2 to bring that group into the picture. This probably would be possible to do. I like the Selmer group associated with a discrete Galois module because it contains more information and also it is easier to relate to the Selmer groups for the specializations.

Concerning the Panchiskin condition, this is kind of a weak form of ordinariness. Panchiskin discovered that with that condition he could construct a p-adic L-function which is "integral." That means that it corresponds to an element theta of the deformation ring R. The p-adic L-function is then of the form phi(theta) as phi varies over the Q_p-valued spectrum of R. (Let me add the remark that it is better to consider the Q_pbar - valued spectrum because, in many cases, the interpolation property is too limited if one restricts to just Q_p-valued phi's.) In any case, I realized that the Panchiskin condition also allows one to define a Selmer group which has a chance of having an interesting (i.e., nonzero) characteristic ideal. That is, the Pontryagin dual of the Selmer group would have a good chance of being a torsion-module over the ring R. The hope is that theta would be a generator of that ideal. Thus, what happens on the algebraic side seems to mirror what happens on the analytic side.

Concerning the non-ordinary case, the difficulties are usually that the natural Selmer group is too big (and doesn't have an interesting characteristic ideal) and, on the analytic side, the p-adic L-function is not integral, i.e. it is not defined in terms of an element theta from R. The simplest example occurs in the theory of elliptic curves and the cyclotomic extension Q_\infty/Q. In the ordinary case, the p-adic L-function should correspond to an element of Lambda. This essentially means that it only has finitely many zeros. In contrast, if E has supersingular reduction at p, then the p-adic L-function has infinitely many zeros. It is much harder, although possible, to formulate a main conjecture in that case.

One final comment is about deformation theory of p-adic Galois representations. You mention this on page 12. You might want to take a look at some of the papers of Mazur. He has developed the theory rather generally, and the ideas have been important. Here are some references:

Deforming Galois representations. Galois groups over Q (Berkeley, CA, 1987), 385--437, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

The theme of p-adic variation. Mathematics: frontiers and perspectives, 433--459, Amer. Math. Soc., Providence, RI, 2000.

Also, there is a long expository article by Fernando Gouvea including many references:

Deformations of Galois Representations," in Arithmetic Algebraic Geometry, ed.

by Brian Conrad and Karl Rubin, American Mathematical Society, 2001, pp. 235--406.

The literature on this topic is rather vast.

I also found a few small misprints. I can send them to you if you want.

Sincerely yours,

Ralph Greenberg

Although they are quite informal and arrived after the rest of the text was written, I also include comments of Laurent Clozel. They are the questions and reservations of an informed specialist and will assure anyone who is troubled by the involuntary imprecision of many assertions in the review that his unease is shared. I have preserved the style of the message as it that customary in Clozel's e-mail communications. I have also responded briefly to the easy queries.

Comments of Laurent Clozel

dear Bob,

i have read your draft in some detail and, although i can't pretend to understand it as well as i would like, i'll send you my remarks.

first some unconnected points. ... 3. i'm writing from home, and after a while i will have to "maintain the connection" which has the effect of garbling the text somehow. i hope it is still readable. 4. i hope you have the state you sent me for i will quote by page & line

now:misprints

...

math

p2, middle : it is not quite clear what ^mu G /M is : it is not a complex group. i assume it is the motivic Galois group of the category generated by M

RPL. Correct!

p3 l4 : "there is ... pi' " : well, not quite if G' is not quasi-split (think of Jacquet-Langlands)

p3 l14 : ^lambda H / pi unique ? it seems to me that it is unique - as the image of the Langlands group - but possibly we cannot determine it by looking at the poles of L-functions?

My first response was that, "there is a pertinent preprint of Song Wang available." This is so, but it is not certain that I understood the question. Clozel replied, “i am aware of these examples, but this was not my point: the point is that (in a world where everything would be known) the image of the Langlands group exists as a well-defined group (up to conjugacy), only we cannot determine what this group is by looking at poles of L-functions. is this incorrect? if not i find your comment confusing ....” This I too believe. It seems to me that we must none the less expect that occasionally the same π will be attached to two or more different nonconjugate homomorphisms into ${}^LG$, perhaps even with images that may be isomorphic but are not conjugate in ${}^LG$. If I am not mistaken, when the images are not connected they may not even be isomorphic. I have not studied the available literature thoroughly.

l.20 ff: here things are not clear to me: first of all the groups ^mu G or ^lambda G are NOT subgroups of G or G' : they are not just reductive groups of finite dimension, but contain the information of their Tannakian categories - they may be subgoups endowed with families of Frobenii, whatever that means. the whole paragraph is cryptic, unless one thinks of the groups as the (motivic or automorphic) groups of a Tannakian subcategory.

p5 top : right and left are inverted in yr diagrams.

p6 l 24 : several parameters: here i don't understand, should'nt there be no L-indiscernability for tori (but this is probably my mistake).

RPL. Local equivalence of automorphic representations implies global equivalence, in particular for tori, but two homomorphisms of the Weil group into the L-group of a torus can be locally equivalent without being globally equivalent.

p6 l -4 ff: here i wonder if you could not pursue the analysis further. first you use the splitting Gal(F/v) --> script T, but in fact you have a splitting of the whole Gal(Q) --> adelic, and in particular Q/p points of script T. (let's take F = Q). assume the field of "coefficients" L is CM. the deformation you seek should be related to this: assume chi is an algebraic Hecke character of L. as such it has exponents (p,q) for all pairs of complex embeddings of L, with p+q = w; let's take w = 0, i.e., anticyclotomic directions. also by Serre, it gives us for each place v, over p, of L, an abelian character of Gal(L). this takes values in a finite extension of L/v; let's assume for simplicity all L/v = Q/p. now we can vary the exponents p (r/2 of them, with the usual notation) in a p-adic continuous way (sorry, p has two meanings). a trivial case of Hida's theory gives us a p-adic family, depending on r/2 parameters, of abelian characters of Gal(L). this looks essentially like a representation Gal(Q) --> finite extension of L/p = product of all local completions of L over p; in fact it will go into the Serre group because we are deforming (p-adically) at w constant. one problem is that this depends on the choice of an initial character (before deformation); the number of choices is a class number. i do not see how to take care of this. the other problem is that we want a deformation of representations of Gal(Q) not of Gal(L), and i do not see how to obviate this. however it may make sense when L is fixed. maybe you can understand how to go further.

p7 l 13 : at this point the reader no longer sees what the "other route" is...

p7, bottom: the variance for T , T^ seems wrong : phi is an analytic one-parameter group to (your) T which should be called T^, so it gives one-parameter groups for (my) T^, p, q, so characters of T where T is in G - correctly, of course, in terms of Harish's theory.

RPL. I hope it is now correct.

p8, middle : distinction even/odd: as Khare tells me it is not true that there are no deformations of an even representation; the problem has been studied by Ramakrishna. in fact there is a deformation space (FINITE over Q/p); what seems to be true is that no deformation will be of geometric (=Fontaine-Mazur) type, except the original one with finite image. (here we encounter the problem, which i had mentioned to you, that you speak rather loosely of "deformations", term which may mean 2 things: deformations in the Taylor-Wiles sense, lifting to char.0, or deformations in the sense of Hida's families of p-adic representations. but i assume the ambiguity is voluntary).

RPL. This is an important correction.

p9, l22 "Sometimes it can be realized" is ambiguous : D can be realized (not G/T).

p10, middle: it is not true that the situation where G(R) is compact has not been studied; in fact the modular deformation theory in this case is done in the C-Harris-Taylor paper you quote, and the p-adic theory is done (for unitary groups) in the thesis of Chenevier.

RPL. This too is an important correction, made by several correspondents.

this concludes my comments: i have read the rest but i do not know enough about p-adic L-functions (amongst other things) to comment. one remark only: p15 l -11 : it is really known (with a suitable definition of the HdR structures) that (6) splits, or is it only natural in the formalism? i will only add that of course your review is very hard to read - more than Hida's book?

regards, l.

In a frivolous vein, I have a final observation. A large number of correspondents were troubled by my designation of the positions on the diagram, to be more precise they thought my use of “left” and “right” incorrect, some even found it grossly negligent. I polled a few acquaintances and family members and found that their use of “left” and “right” was also not mine . So I decided that I should, as a compromise, use “left-hand” and “right-hand”, which would have been less troubling and more precise – or so it seemed at first until I thought of those readers who, for one reason or another, might look at the page upside-down. These two words turned out to be stylistically clumsy as well. In the new text I did my best to find a compromise between my logic, the style, and a desire not to put unnecessary difficulties in the path of a reader.

The imprecision of the adjectives “left” and “right” has troubled me from an early age, especially when referring to the printed page, but I decided when very young that it was the page that determined the sense and not the viewer, whose position could at his pleasure change. This view still seems to me legitimate and applies to all objects that have a definite orientation in space, thus an “up” and a “forward”. A tree does not, but a river, an automobile, a ship (although here “left” and “right” are of course replaced by “port” and “starboard”), a house are generally accepted to determine their own left and right. A stage, too, or a baseball field has its own notion of “left” and “right”. So why not the printed page. I looked for some clear guidance from our household dictionaries, Oxford and Webster, but found little. So a modicum of liberty is still to be found in this one small corner of our intellectual world, but on this occasion I sacrificed it to clarity.

– DRAFT SUBMITTED FOR COMMENTS –

Review of

Haruzo Hida's p-adic automorphic forms on Shimura varieties

by

R.P.Langlands

Three topics figure prominently in the modern higher arithmetic: zeta-functions, Galois representations, and automorphic forms or, equivalently, representations. The zeta-functions are associated to both the Galois representations and the automorphic representations and the link that joins them. Although by and large abstruse and often highly technical the subject has many claims on the attention of mathematicians as a whole: the spectacular solution of at least one outstanding classical problem; concrete conjectures that are both difficult and not completely inaccessible, above all that of Birch and Swinnerton-Dyer; roots in an ancient tradition of the study of algebraic irrationalities; a majestic conceptual architecture with implications not confined to number theory; and great current vigor. Nevertheless, in spite of major results the subject remains inchoate, with far more conjectures than theorems. There is no schematic introduction to it that reveals the structure of both the conjectures whose proofs are its principal goal and the methods to be employed, and for good reason. There are still too many uncertainties. I none the less found while preparing this review that without forming some notion of the outlines of the final theory I was quite at sea with the subject and with the book. So ill-equipped as I am in many ways – although not in all – my first, indeed my major task was to take bearings. The second is, bearings taken, doubtful or not, to communicate them at least to an experienced reader and, in so far as this is possible, even to an inexperienced reader. For lack of time and competence I accomplished neither task satisfactorily. So, although I have made a real effort, this review is not the brief, limpid yet comprehensive, account of the subject, revealing its manifold possibilities, that I would have liked to write and that it deserves. The review is imbalanced and there is too much that I had to leave obscure, too many possibly premature intimations. A reviewer with greater competence, who saw the domain whole and, in addition, had a command of the detail would have done much better.

It is perhaps best to speak of L-functions rather than of zeta-functions and to begin not with p-adic functions but with those that are complex-valued and thus – at least in principle, although one problem with which the theory is confronted is to establish this in general – analytic functions in the whole complex plane with only a very few poles. The Weil zeta-function of a smooth algebraic variety over a finite field is a combinatorial object defined by the number of points on the variety over the field itself and its Galois extension. The Hasse-Weil zeta-function of a smooth variety over a number field F is the product over all places $\mathfrak{p}$ of the inverse of the zeta-function of the variety reduced at $\mathfrak{p}$. Of course, the reduced variety may not be smooth for some $\mathfrak{p}$ and for those some additional care has to be taken with the definition. In fact it is not the Hasse-Weil zeta-function itself which is of greatest interest, but rather factors of its numerator and denominator, especially, but not necessarily, the irreducible factors. The zeta-function is, by the theory of Grothendieck, a product, alternately in the numerator and denominator, of an Euler product given by the determinant of the action $\tau_m(\Phi_{\mathfrak{p}})$ of the Frobenius at $\mathfrak{p}$ on the l-adic cohomology in degree m,

$$\prod_{\mathfrak{p}} \frac{1}{\det\left(1 - \tau_m(\Phi_{\mathfrak{p}})/N\mathfrak{p}^s\right)}. \tag{1}$$

Algebraic correspondences of the variety with itself, if they are present, will act on the cohomology and commute with the Frobenius elements, thereby entailing an additional decomposition of τ_m and an additional factorization of the determinants in (1) and thus of (1) itself. These factors are the L-functions that are one of the key concepts of the modern theory. Grothendieck introduced a conjectural notion of motive, as objects supporting these factors. Although there are many major obstacles to creating a notion of motive adequate to the needs of a coherent theory, not least a proof of both the Hodge and the Tate conjectures, it is best when trying to acquire some insight into the theory's aims to think in terms of motives. In practice they are concrete enough.

So one element of the nexus to be described is the collection $\mathfrak{M}$ of motives M over a given finite extension F of $\mathbb{Q}$. To each M is associated an L-function $L(s, M)$ about which, at first, we know little except that it is an Euler product convergent in a right half-plane. The category of motives as envisioned by Grothendieck is Tannakian so that to each M is also associated a reductive algebraic group ${}^{\mu}G_M$ with a projection onto the Galois group of some sufficiently large, but if we prefer finite, extension K of F. The field of coefficients used for the definition of ${}^{\mu}G_M$ lies, according to needs or inclination, somewhere between $\mathbb{Q}$ and $\bar{\mathbb{Q}}$. The μ in the notation is make it clear that the group has a different function than G. It is not the carrier of automorphic forms or representations but of motives.

An automorphic representation π is a representation, usually infinite-dimensional, of an adelic group $G(\mathbb{A})$, the group G being defined over F and reductive. To G is associated an L-group ${}^LG$, which is a reductive algebraic group over $\mathbb{C}$. There is a homomorphism of ${}^LG$ onto $\mathrm{Gal}(K/F)$, K being again a sufficiently large finite extension of F. Generalizing ideas of Frobenius and Hecke, not to speak of Dirichlet and Artin, we can associate to π and to almost all primes $\mathfrak{p}$ of F a conjugacy class $\{A(\pi_{\mathfrak{p}})\}$ in ${}^LG$. Then, given any algebraic, and thus finite-dimensional, representation r of ${}^LG$, we may introduce the L-function

$$L(s, \pi, r) = \prod_{\mathfrak{p}} \frac{1}{\det\left(1 - r(A(\pi_{\mathfrak{p}}))/N\mathfrak{p}^s\right)}. \tag{2}$$

The usual difficulties at a finite number of places are present.

In principle, and in practice so far, the functions (2) are easier to deal with than (1). Nevertheless, the initial and fundamental question of analytic continuation is still unresolved in any kind of generality. One general principle, referred to as *functoriality* and inspired by Artin's reciprocity law, would deal with the analytic continuation for (2). Functoriality is the core notion of what is frequently referred to as the Langlands program.

Suppose G and G' are two groups over F and ϕ is a homomorphism from ${}^LG$ to ${}^LG'$, then if π is an automorphic representation of G there is an automorphic representation π' of G' such that for each $\mathfrak{p}$ the class $\{A(\pi'_{\mathfrak{p})}\}$ associated to π' is the image under ϕ of $\{A(\pi_{\mathfrak{p})}\}$. To establish this will be hard and certainly not for the immediate future. I have, however, argued in [L] that it is a problem that we can begin to attack.

It is then natural to suppose, once again influenced by Artin's proof of the analytic continuation of abelian L-functions, that each of the Euler products $L(s, M)$ into which (1) factors is equal to one of the Euler products (2). This would of course certainly deal with the problem of its analytic continuation. Better, in [L] it is suggested that we should not only prove functoriality using the trace formula but simultaneously establish that each automorphic representation π on G is associated to a subgroup ${}^\lambda H_\pi$ of ${}^LG$, even to several such subgroups, but the need for this multiplicity is something easy to understand. So we are encouraged to believe that the fundamental correspondence is not that between L-functions but that between M and ${}^\mu G_M$ and π and ${}^\lambda H_\pi$. In particular ${}^\mu G_M$ and ${}^\lambda H_\pi$ are to be isomorphic and the Frobenius-Hecke conjugacy classes in ${}^\mu G_M$ associated to M are to be equal to the Frobenius-Hecke conjugacy classes ${}^\lambda H_\pi$ associated to π. Apart from the difficulty that there is little to suggest that ${}^\lambda H_\pi$ is defined over any field but $\mathbb{C}$, it is reasonable to hope that in the long run some correspondence of this nature will be established. The λ in the notation is inherited from [L] and emphasizes that H is a subgroup of ${}^LG$ and not of G.

The Tannakian formalism for motives – when available – suggests that if there is a homomorphism ${}^\mu G_M \subset {}^\mu G'$ then M is also carried by ${}^\mu G'$. If functoriality is available, as is implicit in the constructions, and ${}^\lambda H_\pi \subset {}^\lambda H'$ then, in some sense, π is also carried by ${}^\lambda H'$, but in the form of an automorphic representation π' of a group G' with ${}^\lambda H' \subset {}^LG$. So if ${}^\mu G'$ and ${}^\lambda H'$ are isomorphic, the couples $\{M, {}^\mu G'\}$ and $\{\pi, {}^\lambda H'\}$ also correspond.

An example, in spite of appearances not trivial, for which the necessary functoriality is available is the unique automorphic representation π of the group $G = \{1\}$ with ${}^LG = {}^\lambda H_\pi = \mathrm{Gal}(L/F)$, where $\mathrm{Gal}(L/F)$ is solvable, together with the motive $M(\sigma)$ of rank 2 and degree 0 attached to a faithful two-dimensional representation σ of $\mathrm{Gal}(L/F)$. They clearly correspond. Moreover ${}^\lambda H_\pi = {}^\mu G_M$ is imbedded diagonally in $GL(2,\mathbb{C})\times\mathrm{Gal}(L/F)$. The representation π' is given by solvable base-change and the correspondence between $\{\pi', GL(2,\mathbb{C})\times\mathrm{Gal}(L/F)\}$ and $\{M(\sigma), GL(2,\mathbb{C})\times\mathrm{Gal}(K/F)\}$ is one of the starting points for the proof of Fermat's theorem.

Although functoriality and its proof are expected to function uniformly for all automorphic representations, when comparisons with motives are undertaken not all automorphic representations are pertinent. The representation π has local factors π_v at each place. At an infinite place v the classification of the irreducible representations π_v of $G(F_v)$ is by homomorphisms of the Weil group at v into ${}^LG$. This Weil group is, I recall, a group that contains $\mathbb{C}^\times$ as a subgroup of index 1 or 2. We say ([Ti]) that the automorphic representation π is *arithmetic* (or algebraic or motivic) if for each place π_∞ is parametrized by a homomorphism whose restriction to $\mathbb{C}^\times$, considered as an algebraic group over $\mathbb{R}$, is itself algebraic. Thus it is expressible in terms of characters $z \in \mathbb{C}^\times \to z^m \bar{z}^n$, $m, n \in \mathbb{Z}$.

Only arithmeticic automorphic representations should correspond to motives. Thus the second element of our nexus is to be the collection $\mathfrak{A}$ of automorphic representations π for F, each attached to a group ${}^\lambda H$. Because of functoriality, in the stronger form described, π is no longer bound to any particular group G.

A central problem is to establish a bijective correspondence between the two elements introduced. Major progress was made by Wiles in his proof of the conjecture of Taniyama and Shimura. Since he had – and still would have – only an extremely limited form of functoriality to work with, his arguments do not appear in exactly the form just suggested. Moreover, there are two further extremely important elements in the nexus in which he works to which we have not yet come.

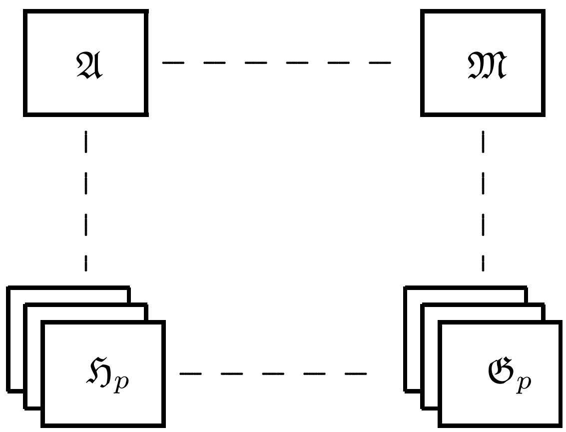
To each motive M and each prime p is attached a p-adic representation of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ of dimension equal to the rank of the motive. The third element of the nexus is not, however, the collection of p-adic Galois representations – subject to whatever constraints are necessary and appropriate. Rather it is a foliated space, in which the leaves are parametrized by p and in which there are passages from one leaf to another, permitted in so far as each p-adic representation is contained in a compatible family of representations, one for each prime. We are allowed to move from one leaf to another provided we move from one element of a compatible family to another element of the same family. The arguments of Wiles and others, those who preceded and those who followed him, rely on an often very deep analysis of the connectivity properties of the third element, either by p-adic deformation within a fixed leaf, in which often little more is demanded than congruence modulo p, or by passage from one leaf to another in the way described (cf. [Kh]) and their comparison with analogous properties of yet a fourth element whose general definition appears to be somewhat elusive.

For some purposes, but not for all, it can be taken to consist of representations of a suitably defined Hecke algebra. For automorphic representations attached to the group G over F, the Hecke algebra is defined in terms of smooth, compactly supported functions f on $G(\mathbb{A}_F^f)$, $\mathbb{A}_F^f$ being the adèles whose components at infinity are 0. They act by integration on the space of any representation π of $G(\mathbb{A}_F)$, in particular on the space of an automorphic representation or on automorphic forms.

Let $\mathbb{A}_F^\infty$ be the product of F_v at the infinite places. When the Lie group $G(\mathbb{A}_F^\infty)$, defines a bounded symmetric domain – or more precisely when a Shimura variety is attached to the group G – then there are quotients of the symmetric domain that are algebraic varieties defined over number fields. There are vector bundles defined over the same field whose de Rham cohomology groups can be interpreted as spaces of automorphic forms for the group G on which the Hecke operators will then act. The images of the Hecke algebra will be finite-dimensional algebras over some number field L and can often even be given an integral structure and then, by tensoring with the ring $\mathcal{O}_\mathfrak{p}$ of integral elements at a place $\mathfrak{p}$ of L over p, a p-adic structure, imparted of course to its spectrum. In so far as these rings form the fourth element of the nexus, the leaves are clear, as is the passage from one leaf to another. It seems to correspond pretty much to taking two different places $\mathfrak{p}$ and $\mathfrak{q}$ without changing the homomorphism over L.

The four elements form a square, motives at the top left, automorphic representations at the top right, the leaves $\mathfrak{G}_p$ of the p-adic representations at the bottom left, and the fourth as yet only partly defined element $\mathfrak{H}_p$ at the bottom right. The heart of the proof of Fermat's theorem is to deduce from the existence of one couple $\{M, {}^{\mu}G_M\} \in \mathfrak{M}$ and $\{\pi, {}^{\lambda}H_\pi\} \in \mathfrak{A}$ of corresponding pairs the existence of others. We pass from $\{M, {}^{\mu}G_M\}$ in $\mathfrak{M}$ to some leaf in the element below, thus to the corresponding p-adic Galois representation $\mathfrak{s}_p \in \mathfrak{G}_p$, and from $\{\pi, {}^{\lambda}H_\pi\}$ to an object $\mathfrak{h}_p \in \mathfrak{H}_p$, the fourth element of the nexus. Then the essence of the arguments of Wiles and Taylor-Wiles is to show that movement in $\mathfrak{G}_p$ of the prescribed type is faithfully reflected in permissible movements in $\mathfrak{H}_p$ and that if in $\mathfrak{G}_p$ the movement in leads to an image of a pair in M then the corresponding movement in $\mathfrak{H}_p$ leads to an element of $\mathfrak{A}$. These two pairs will then necessarily correspond in the sense that the associated Frobenius-Hecke classes will be the same.

As a summary of the proof of Fermat's theorem, the preceding paragraph is far too brief, but it places two features in relief. There has to be an initial seeding of couples with one term from $\mathfrak{M}$ and one from $\mathfrak{A}$ that are known for some reason or another to correspond and it has to be possible to compare the local structures of the two spaces $\mathfrak{G}$ and $\mathfrak{H}$.



The easiest seeds arise for G an algebraic torus, for then an automorphic representation π is a character of $T(\mathbb{A}_F)$ and if the character is of type A_0, thus if the representation is arithmetic, the process begun in [W] and continued by the construction of the Taniyama group ([LS]), should construct both the p-adic representations and the motive $\{M, {}^{L}T\}$ corresponding to $\{\pi, {}^{L}T\}$. From them others can be constructed by functoriality, a formality for M.

Although they are somewhat technical, it is useful to say a few words about the correspondence for tori, partly because it serves as a touchstone when trying to understand the general lucubrations, partly because the Taniyama group, the vehicle that establishes the correspondence between arithmetic automorphic forms on tori and motives, is not familiar to everyone. Most of what we need about it is formulated either as a theorem or as a conjecture in one of the papers listed in [LS], but that is clear only on close reading. In particular, it is not stressed in these papers that the correspondence yields objects with equal L-functions

The Taniyama group as constructed in the first paper of [LS] is an extension $\mathcal{T} = \mathcal{T}_F = \varprojlim_L \mathcal{T}_F^L$ of the Galois group $\mathrm{Gal}(\bar{F}/F)$, regarded as an pro-algebraic group, by a pro-algebraic torus $\mathcal{S} = \varprojlim_L \mathcal{S}_F^L$ and is defined for all number fields F finite over $\mathbb{Q}$. One of its distinguishing features is that there is a natural homomorphism φ_F of the Weil group W_F of F into $\mathcal{T}_F(\mathbb{C})$. This homomorphism exists because there is a splitting of the image in $\mathcal{S}_f^L(\mathbb{C})$ of the lifting to $W_{L/F}$ of the Galois 2-cocycle in $H^2(\mathrm{Gal}(L/F), \mathcal{S}_f^L(\bar{\mathbb{Q}})$ defining $\mathcal{T}_F^L$. Thus every algebraic homomorphism over $\mathbb{C}$ of $\mathcal{T}_F$ into an L-group ${}^LG$ compatible with the projections on the Galois groups defines a compatible homomorphism of W_F into ${}^LG(\mathbb{C})$. In particular if $G = T$ is a torus, every T-motive over $\mathbb{C}$ (if all conjectures are anticipated, this is just another name for a homomorphism ϕ from $\mathcal{T}_F$ to ${}^LT$) defines a homomorphism $\psi = \varphi \circ \phi$ of the Weil group into ${}^LT$ and thus ([LM]) an automorphic representation π of $T(\mathbb{A}_F)$.

The Weil group can be constructed either at the level of finite Galois extensions L/F as $W_{L/F}$ or as a limit W_F taken over all L. The group $W_{L/F}$ maps onto the Galois group $\mathrm{Gal}(L^{\mathrm{ab}})$. The kernel is the closure of the image of $I_L^\infty = \prod_{v|\infty} L_v^\times$. A key feature of the construction and, especially, of the definition of the group $\mathcal{S}$ that permits the introduction of φ_F is the possibility of constructing certain elements of the group of idèles I_L well-defined modulo the product of I_L^∞ with the kernel of any given continuous character. Moreover in the construction an imbedding of $\bar{\mathbb{Q}}$ in $\mathbb{C}$ is fixed, so that the collection of imbeddings of L in $\mathbb{C}$ may be identified with $\mathrm{Gal}(L/\mathbb{Q})$ or, if the imbedding of F is fixed, with $\mathrm{Gal}(L/F)$. The automorphic representation π associated to ϕ will be arithmetic because of the definition of the group $X^*(\mathcal{S})$ of characters of $\mathcal{S}$ and because of the definition of φ_F.

Conversely every arithmetic automorphic representation π of T arises in this way. Such a representation is attached (cf. [LM]) to a parameter, perhaps to several, $\psi : W_{L/F} \to {}^LT(\mathbb{C})$. The field L is some sufficiently large but finite Galois extension of F. If π is arithmetic this parameter factorizes through φ_F. To verify this, take L so large that all its infinite places are complex and observe first of all that ψ restricted to the idèle classe group $C_L = I_L/L^\times$ defines a homomorphism of $I_L^\infty \subset I_L$ to $\mathcal{S}^L$ and for any character λ of T, there is a collection of integers $\{\lambda_\tau \mid \tau \in \mathrm{Gal}(L/F)\}$ such that

$$\lambda(\psi(x)) = \prod_{\tau \in \mathrm{Gal}(L/F)} \tau(x)^{\lambda_\tau}.$$

The function $\lambda \to \lambda_\tau$ is a character of $\mathcal{S}^L$ and it defines the homomorphism ϕ from $\mathcal{S}^L$ to $\hat{T}$. To extend it to $\phi : \mathcal{T}^L \to {}^LT$ all we need do is split the image in $\hat{T}(L^{\mathrm{ab}})$ under ϕ of the cocycle in $H^2(\mathrm{Gal}(L^{\mathrm{ab}}/F), \mathcal{S}^L)$ defining $\mathcal{T}^L$ with the help of ψ. If $w \in W_{L/F}$ maps to τ in $\mathrm{Gal}(L^{\mathrm{ab}}/F)$ and to $\bar{\tau}$ in $\mathrm{Gal}(L/F)$ then $a(\tau)$, the representative of τ in $\mathrm{Gal}(L^{\mathrm{ab}}/F)$ used in the first paper of [LS] to define $\mathcal{T}_F^L$, then $\phi(a(\tau)) = \phi^{-1}(a(\tau)^{-1}\varphi_F(w))\psi(w)$. The right side is well-defined because of the definition of the groups $\mathcal{S}^L$ and $\mathcal{T}^L$.

As emphasized in the first paper of [LS], for each finite place v of F there is a splitting $\mathrm{Gal}(\bar{F}_v/F_v) \to \mathcal{T}^L(F_v)$, thus a v-adic representation of $\mathrm{Gal}(\bar{F}_v/F_v)$ in $\mathcal{T}^L$, in particular a p-adic representation if $F = \mathbb{Q}$ and $v = p$. At the moment, I do not understand how or under what circumstances this representation can be deformed and I certainly do not know which, if any, of the general conjectures about L-values and mixed motives to be described in the following pages are easy for it, which are difficult, or which have been proved (cf. [MW,R]).

The toroidal seeds themselves will be, almost without a doubt, essential factors of any complete theory of the correspondence between motives and arithmetic automorphic forms. There are two conceivable routes: either attempt to establish and use functoriality in general or attempt to use only the very little that is known about functoriality at present but to strengthen the other, less analytical and more Galois-theoretic or geometric parts of the argument. Although functoriality in general is not just around the corner, it is a problem for which concerted effort now promises more than in the past. So there is something to be said for reflecting on whether it will permit the correspondence between $\mathfrak{A}$ and $\mathfrak{M}$ to be established in general. I stress, once again, that up until now only simple seeds have been used, perhaps only those for which the group T is the trivial group $\{1\}$.

The principal merit of the second route is perhaps that it quickly confronts us with a difficulty carefully skirted in the above presentation, an adequate definition of the fourth element $\mathfrak{H}$. In addition, starting with known couples, the method can also arrive at other couples, of which the first element, thus the element in $\mathfrak{A}$ can, because of the element in $\mathfrak{M}$ with which it is paired, be identified with the functorial image of a representation of a second group. Such examples are a feature of the work of Richard Taylor and his collaborators ([Ta], but see also [Ki]) on odd icosahedral representations or on the Sato-Tate conjecture. Although their present forms were suggested by functoriality, these problems are of great independent interest and can be presented with no reference to it – and sometimes are. Nevertheless, functoriality is expected to be valid for all automorphic representations, not just for arithmetic automorphic representations, and is indispensable for analytic purposes such as the Selberg conjecture. So proofs of it that function only in the context of arithmetic automorphic representations are not enough.

I have so far stressed the correspondence between the four elements $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{G}$ and $\mathfrak{H}$ partly because the research of the most popular appeal as well as much of the wave that arose in the wake of the proof of Fermat's theorem involves them all. There is nevertheless a good deal to be said about the relation between the deformations in $\mathfrak{G}_p$ and those in $\mathfrak{H}_p$ that bear more on the structure of the elements of $\mathfrak{M}$ and on the problematic definition of $\mathfrak{H}$ than on the relation between $\mathfrak{M}$ and $\mathfrak{G}$. The notion of a deformation in $\mathfrak{H}$ or $\mathfrak{H}_p$ remains imprecise and it is not at first clear when two elements of $\mathfrak{H}$ or $\mathfrak{G}$ are potentially in the same connected component. By definition there is associated to each arithmetic automorphic representation a family $\{\varphi_v\}$, v running over the real infinite places, of homomorphisms of the Weil group $W_{\mathbb{C}/\mathbb{R}}$ into an L-group ${}^LG$. The restriction of φ_v to $\mathbb{C}^\times$ can be assumed to have an image in any preassigned Cartan subgroup T of the connected component $\hat{G}$ of ${}^LG$ and will be of the form $z \to z^\lambda \bar{z}^\mu$, where $\lambda, \mu \in X^*(T)$ are characters of T. The homomorphism φ_v is then determined by a choice of w in the normalizer of T in L^G of order two modulo T itself whose image in the Galois group is complex conjugation at v and which satisfies $w^2 = e^{\pi i(\lambda-\mu)}$, $w\lambda = \mu$. Since φ_v is only determined up to conjugation, there are equivalence relations on the triples $\{w, \lambda, \mu\}$, but the essential thing is that for each real v the homomorphisms fall into families, defined by w and the linear space in which λ lies. These spaces may intersect, and in the intersection there is ambiguity. For example, if the ground field is $\mathbb{Q}$, $G = GL(2)$, ${}^LG = GL(2, \mathbb{C})$ and

$$T = \{t(a,b)\} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\}$$

then the action of w on T is either trivial or it takes $t(a,b)$ to $t(b,a)$. Moreover

$$z^\lambda \bar{z}^\mu = t(z^k \bar{z}^l, z^m \bar{z}^n)$$

with $k = l$, $m = n$ if the action is trivial and $k = n$, $l = m$ if it is not. In the first case, $w^2 = 1$, so that w can be taken as $t(\pm 1, \pm 1)$. In the second, at least if $k \neq l$, w can be taken in the form

$$w = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \quad \alpha = e^{\pi i(k-l)}. \tag{3}$$

This is a possibility even if $k = l$. It is equivalent to the particular choice $w = \pm t(1, -1)$. Thus, even though the parameters λ and μ are discrete and not continuous it is natural to distinguish two components in the space of parameters. In each λ is arbitrary and $\mu = w\lambda$, but in the first $w = \pm t(1,1)$ and in the second w is given by (3). These two families reappear in $\mathfrak{G}$ as even and odd Galois representations, the odd being apparently readily deformable, while the even seem to admit at best trivial deformations, as happens for reducible representations. There are similar families for other groups. Formally the number of parameters will be the dimension of the space of λ, thus the rank of the group G.

Although deformation in these parameters is not possible in $\mathfrak{A}$ or $\mathfrak{M}$, the deformations in $\mathfrak{G}_p$ or $\mathfrak{H}_p$ appear in some sense as deformations within families like those just described. Nevertheless the most important step in the proof of Wiles is a comparison of the local structure of $\mathfrak{H}_p$ and $\mathfrak{G}_p$ that does not involve a variation of the parameter, which we should think of as a Hodge type, or rather as the source of the Hodge type, motives being objects that are realized by a linear representation of the associated group, and the Hodge type being affected by the realization. The parameter maps to the L-group ${}^LG$, so that representations r of ${}^LG$ are an important source of realizations.

It is the deformations within $\mathfrak{H}_p$ and their structure that are central to much of Hida's efforts over the past two decades. His early work on the infinitesimal structure of $\mathfrak{H}_p$ for modular forms or for Hilbert modular forms appears to my untutored eye to have been a serious influence, but of course by no means the only one, on developments that ultimately led to a proof of Fermat's theorem.

As the notation indicates the spaces $\mathfrak{H}_p$ are related to Hecke algebras, but these algebras cannot be exactly those that are defined by the algebra of compactly supported functions on $G(\mathbb{A}_f)$ acting on automorphic forms, thus on complex-valued functions on $G(F)\backslash G(\mathbb{A}_F)$, because the algebras defining H_p must be algebras over a number field or, at least, over an extension of $\mathbb{Q}_p$.

For classical automorphic forms or representations the difficulty is not so egregious, since from the subject's very beginning the modular curves were present. The forms appeared as sections of line bundles on them, so that a structure of vector space over $\mathbb{Q}$ or some other number field or of module over $\mathbb{Z}$ was implicit in their very definition. In general, however, an adequate definition of $\mathfrak{H}_p$ remains problematic.

Suggestions can be made. It is natural to look for actions of the Hecke operators on cohomology groups, for these can be taken either in the Betti form so that they have a $\mathbb{Q}$-structure or, if the group G defines a Shimura variety, in the de Rham form so that they are defined over a number field. Not only can both, or at least their tensor products by $\mathbb{R}$ or $\mathbb{C}$, be calculated in representation-theoretic terms ([BW,GS]) but there are general theorems to compare the two, of which I suppose the Eichler-Shimura map favored by Hida is a particular manifestation. Although he intimates both possibilities before finally favoring a presentation in the de Rham form, Hida does not undertake a description of the general background. A brief but thorough account of it would have been of great benefit to the reader, the reviewer, and perhaps to the author as well.

To continue, I take, by restriction of scalars if necessary, the group G to be defined for simplicity over $\mathbb{Q}$. Suppose, as in [GS], that B is a Borel subgroup of G over $\mathbb{C}$ and that $B(\mathbb{C}) \cap G(\mathbb{R})$ is a Cartan subgroup $T(\mathbb{R})$ of $G(\mathbb{R})$ whose projection on the derived group is compact. Then $B(\mathbb{C})\backslash G(\mathbb{C})$ is a projective variety and the complex manifold $F = T(\mathbb{R})\backslash G(\mathbb{R})$ is imbedded in it as an open subset. If K_∞ is a maximal compact subgroup of $G(\mathbb{R})$ containing $T(\mathbb{R})$ then $T(\mathbb{R})\backslash G(\mathbb{R})$ is a fiber space over $D = K_\infty\backslash G(\mathbb{R})$. Sometimes D can be realized as a bounded symmetric domain and then for any open compact subgroup K_f of $G(\mathbb{A}^f)$ the complex manifold $D \times G(\mathbb{A}^f)/K_f$ can carry the structure of a Shimura variety, whose exact definition demands a little additional data that it is not useful to describe here. Of paramount importance, however, is that the variety is defined over a specific number field, the reflex field, and that if K_f is sufficiently small it is smooth, although not necessarily complete.

In particular, if the adjoint group of $G(\mathbb{R})$ is compact then D is a point and trivially a bounded symmetric domain. This is perhaps significant because the arguments of, for example, [T,Ki,Kh], not to speak of those in §4.3 of the book under review, often appeal to the JL-correspondence, at least in the special case of $GL(2)$, but because of the recent work by Laumon and Ngo on the fundamental lemma a proof of the correspondence, a special, comparatively easy case of functoriality, is – with time and effort – within reach for all groups. The correctly formulated correspondence relates automorphic representations on a group and an inner twisting of it and any group over $\mathbb{R}$ with a compact Cartan subgroup has an inner twisting that is compact.

Any character of T defines a line bundle on F, but also a cocharacter of type A_0 of a Cartan subgroup of the connected component $\hat{G}$ of ${}^LG$. The cocharacter can be extended to a homomorphism of the Weil group $W(\mathbb{C}/\mathbb{R})$ and this homomorphism defines a parameter ϕ_∞ and an L-packet of representations in the discrete series of $G(\mathbb{R})$; moreover, according to [GS] a substantial part of the cohomology of the line bundle is yielded by the automorphic forms associated to these L-packets. For a given group, just as for the example of $GL(2)$, it appears that all these parameters are expected to define the same connected component of $\mathfrak{H}_p$ or $\mathfrak{G}_p$.

Although Hida recognizes clearly the need for general definitions of $\mathfrak{H}_p$, he concentrates on groups G that define a Shimura variety. As their designation suggests these varieties were introduced and studied by Shimura in a long series of papers. Although their importance was quickly recognized, these papers were formulated in the algebro-geometric language created by Weil, not in the more supple and incisive language of Grothendieck, especially suited to moduli problems, and were not easily read. An influential Bourbaki report by Deligne in 1971 clarified both the basic definitions and the proofs, although he like Shimura only treated those varieties, a very large class, which are essentially solutions of moduli problems. The remaining varieties were eventually treated by Borovoi ([BM]) by different methods. Neither the papers of Borovoi nor the investigations that preceded them are mentioned by Hida and the reader of his book is strongly advised to turn elsewhere for an introduction to the modern theory of Shimura varieties, for example to the lectures of Milne ([BM]). For the purposes of the book, only special Shimura varieties are needed, but that is presumably a reflection of the limitations of current methods.

At present, to give any definition whatsoever of $\mathfrak{H}_p$, one has either to work with groups with $G(\mathbb{R})$ compact modulo its center or with groups for which the associated Shimura varieties can be defined over the ring of integers in some finite extension of $\mathbb{Q}$. The first possibility has not, so far as I know, been examined, except in some low-dimensional cases, and the second requires, for the moment, that the variety be the solution of a moduli problem. Then H_p is the algebra of Hecke operators acting on p-adic automorphic forms. Classically these automorphic forms had been investigated by others earlier (cf.[KS]), but Hida discovered even for classical forms some remarkable features that seem to appear for general groups as well.

The present book is an account of that part of the theory developed by him for several important types of Shimura varieties: modular curves, Hilbert modular varieties, and Siegel modular varieties. The publishers recommend it as a text for graduate students, but that is irresponsible. Although many of Hida's early papers and a number of his books are very well written, neither expository flair nor a pedagogical conscience are evident in the present text. The style is that of rough lecture notes, cramped pages replete with formulas and assertions that run one into the other, largely obscuring the threads of the argument, and with an unchecked flood of notation. The meaning of essential symbols is variable and not always transparent so that the reader is occasionally overcome by a disconcerting uncertainty.

On the other hand, Hida's goals, both those realized in the book and those still unrealized, are cogently formulated in his introduction and, so far as I can appreciate, of considerable interest. Experts or even experienced mathematicians in neighboring domains, for example the reviewer, will I believe be eager to understand his conclusions, but they, and the author as well, might have been better served either by a series of normal research papers or by a frankly pedagogical monograph that assumed much less facility with the technical apparatus of classical and contemporary algebraic geometry. The material is difficult and in the book the definitions and arguments come at the reader thick and fast, in an unmitigated torrent in which I, at least, finally lost my footing.

Although as I have emphasized, the parameters ϕ_∞ of the discrete series or of the Hodge structure seem to lie in the same connected component, we cannot expect to pass continuously from one to another. Indeed, for Shimura varieties, fixing the parameter corresponds approximately to fixing the weight of the form. The dimension of the space of automorphic forms being, according to either the trace formula or the Riemann-Roch formula, pretty much a polynomial in λ, we cannot expect it to be constant and independent of λ. For p-adic forms, however, there are large families of constant dimension that interpolate, in a space with p-adic parameters, a certain class of arithmetic automorphic forms.

The Hecke algebra and its actions are just another expression of the automorphic representations or of the automorphic forms. Fixing imbeddings of $\bar{Q}$ into $\mathbb{C}$ and into $\bar{\mathbb{Q}}_p$ and taking all fields F to be subfields of $\bar{Q}$, at a finite place p we replace the collection of local parameters $\phi_\infty = \{\phi_v\}$, $v|\infty$, by a collection of homomorphisms of the local Weil groups W_{F_v}, into the L-group over $\mathbb{C}$. For those representations $\pi = \otimes\pi_v$ that are associated to motives, these parameters will presumably be given by homomorphisms σ_v, $v|p$, of the Galois groups $\mathrm{Gal}(\bar{F}_v/F_v)$, $v|p$, into the L-group over $\bar{Q}_l$, where l may or may not be equal to p.

If $l \neq p$, such a homomorphism will be tamely ramified and the restriction to the decomposition group is strongly limited and does not offer much room for deformation. It may as well be fixed, so that the deformations will take place over the image of the Frobenius which there seems to be no attempt to constrain. If, however, $p = l$, the possibilities for the σ_v are at first manifold but when the representations σ_v, $v|p$, arise from a motive they are constrained in an important way first discovered by Tate. They can be assigned a Hodge-Tate type whose basic description in terms of parameters λ subject to an integrality condition is much like that attached to the Hodge structures at infinity. Since Tate's paper [T] a very great deal has been learned about the restrictions of the p-adic representations associated to motives to the decomposition groups at places v dividing p ([FI]) that appears to be indispensable for the study the spaces $\mathfrak{H}_p$ or $\mathfrak{G}_p$, but what the reader of the present book will discover is that at p the Hodge type seems to control the possible deformations just as it did at infinity in combination with the elements w of order two. In the much studied case of the group $GL(2)$, a w with the two eigenvalues $+1$ and -1 can allow many deformations but a w with equal eigenvalues does not appear to do so. At p the analogous dichotomy seems to be between ordinary and extraordinary or – more colloquially expressed – nonordinary, although I suppose that there will ultimately be a whole spectrum of possibilities each permitting some kinds of deformation and forbidding others. The ordinary case is presumably the optimal case and is the one on which Hida concentrates.

For the types at ∞ there was no possibility of real deformation because λ was constrained by an integrality condition. At p it is possible to abandon the integrality condition because the decomposition group of the infinite cyclotomic extension $\mathbb{Q}_{\mu_{p^\infty}}$ is $\mathbb{Z}_p^\times$ which is isomorphic to the product of the group $\mathbb{F}_p^\times$ with $1 + p\mathbb{Z}_p$ and the second factor admits a continuous family of characters $x \to x^a$, $a \in \mathbb{Z}_p$, interpolating the characters given by integral a. This allows for deformation or interpolation in the space $\mathfrak{G}_p$ which is, it turns out, accompanied by possible deformations in the space $\mathfrak{H}_p$. The new parameter is usually not just an open subset of $\mathbb{Z}_p^\times$ but, as for abelian G, of some subspace of $X^*(T) \otimes \mathbb{Z}_p$, $X^*(T)$ being the space of characters of a Cartan subgroup of G.

This discovery by Serre (cf. [KM]), whose work was followed by that of Katz and preceded by that of Swinnerton-Dyer can perhaps be regarded as a second point where Ramanujan influenced the course of the general theory of automorphic forms in a major way, for Swinnerton-Dyer was dealing with congruences conjectured by him. The first point was of course the Ramanujan conjecture itself, which led, through Mordell and Hecke, to the general theory of automorphic L-functions. Hida appreciated that in the p-adic theory, where the weight was no longer integral, there was a possibility of the uniform deformation of whole families of modular forms, the ordinary forms, to a rigid-analytic parameter space, thus to an open subset of $\mathbb{Z}_p^n$ for some integer n. It would be surprising if this possibility were limited to $GL(2)$ and Hida has devoted a great deal of time, energy and space to the admirable design of creating a general theory. To read his books and papers grows increasingly difficult; to read them alone without consulting those of other authors, Katz or Fontaine for example, or, in a different optic, Taylor or Khare, is ill-advised, even impossible for some of us. Nevertheless, although no-one, neither Hida nor anyone else, appears to have broken through to a clear and comprehensive conception of the ultimate theory, there is a great deal to be learnt from his writings, both about goals and about techniques. In spite of Hida's often trying idiosyncrasies, to follow his struggles for a deep and personal understanding of the resistant material is, as Tilouine observed in a briefer review, not only edifying but also challenging, although it appears to be easier to begin with the earlier papers, for they are often more concrete and in them some key ideas are less obscured by technical difficulties and general definitions.

Hida has also been preoccupied with two problems parallel to that of constructing deformations of p-adic forms: parametrized families of p-adic Galois representations; p-adic L-functions. Although the theory of parametrized families of Galois representations is not developed in the book under review and, indeed, so far as I know, unless very recently, has hardly been developed beyond $GL(2)$, it is adumbrated in the introduction as one of the ultimate goals of the author. In earlier papers of Hida ([Hi]), the elaborate "infinitesimal" structure, whose appearance in $\mathfrak{H}_p$ is for $GL(2)$ a manifestation of congruences between the Fourier expansions of automorphic forms and whose coupled appearance in $\mathfrak{H}_p$ and $\mathfrak{G}_p$ is a key feature of the proof of Fermat's theorem, appears and is investigated not only for fixed weight and central character but also for entire parametrized families. There is much more number-theoretical information in these investigations than I have been able to digest.

The elements of $\mathfrak{A}$ or of $\mathfrak{H}_p$ are attached to automorphic representations or forms, thus to a particular group G and to a particular L-group ${}^LG$, but to the extent that functoriality is available the group G can be replaced by others G' and the representation π of $G(\mathbb{A}_F)$ by another π' of $G'(\mathbb{A}_F)$. The p-adic Galois representations can be modified in the same way, and without any ado. It might be worth reflecting on how the passage to the primed objects should be interpreted in $\mathfrak{H}_p$.

A final, major goal described briefly in the introduction to the book and of concern to many people (*cf.* [Gr]) is the construction of p-adic L-functions. They seem to me of such importance both to Hida's project and to all mathematicians with an interest in number theory that I cannot end this review without a very brief and even more superficial description of the attendant questions. I have no clear idea of their current state. I believe that we can safely assume that they are largely unanswered.

The complex L-functions attached to $\pi \in \mathfrak{A}$ or to a motive $M \in \mathfrak{M}$ are specified only when in addition a finite-dimensional complex representation r of ${}^LG$ is given, π being an automorphic representation of G and M a motive of type ${}^LG$. It is of the form $L(s, \pi, r)$ or $L(s, M, r)$ although both can be – in principle! – written as $L(s, \pi')$ or $L(s, M')$, $\pi' = \pi_r$ an automorphic representation of $GL(n)$, $M' = M_r$ a motive of rank d, $d = \dim \rho$. Of course, if M is associated to π then $L(s, \pi, r) = L(s, M, r)$. These somewhat speculative remarks are meant only to emphasize that all problems related to the p-adic L-functions will have to incorporate r. They will also have to incorporate the parameter space of the deformations, which appears to be, the elaborate local structure aside, at its largest, an open subset A of $X^*(T) \otimes \mathbb{Z}_p$, T being a Cartan subgroup of G over the chosen ground field F, but it is of this size only in unusual situations. As we noticed for tori, there are important constraints on the subspace in which A is to be open. $X^*(T) \otimes \mathbb{Z}_p$ has of course a Galois action.

The functions are to be p-adic analytic functions $L_p(s, r)$ on the parameter space, thus on a subset of $\mathfrak{H}_p$ or $\mathfrak{G}_p$ identified with the set A in $X^*(T) \otimes \mathbb{Z}_p$. Elements $s = \mu \times z$ of this space define equivariant homomorphisms of open subgroups of $K \otimes \mathbb{Z}_p$ into $\hat{T}(\mathbb{Q}_p)$ in the form $a \to \prod_{\varphi|p} \varphi(a)^{z\varphi(\mu)}$. Moreover at points in $\lambda \in X * (T) \cap A$ (or at least at a large subset of this space, perhaps defined by a congruence condition) the element of $\mathfrak{G}_p$ is to be the image of a motive $M(\lambda)$. So $M'(\lambda) = M_r(\lambda)$ is defined. The p-adic function $L_p(s, r)$ is to interpolate in an appropriate form values $R(M'(\lambda))$ of the complex L-functions $L(z, M'(\lambda))$ at $z = 0$.

There are many important conjectures pertinent to the definition of $R(M'(\lambda))$. Unfortunately we do not have the space to describe them fully ([Mo,Ha]), but something must be said. For this it best to simplify the notation and to suppose $M = M'(\lambda)$. When discussing the L-function $L(z, M)$ it is best to suppose that M is pure, thus that all its weights are equal, for otherwise there is no well-defined critical strip, and no well-defined center. Since every motive will have to be a sum of pure motives, this in principle presents no difficulty.

Motives are defined (in so far as they are well-defined) by projections constructed from linear combinations of algebraic correspondences with coefficients from a field K of characteristic zero. It is customary to take K to be a finite extension of $\mathbb{Q}$. The field K is not the field over which the correspondences are defined. That field is F, the base field. It is probably best, for the sake of simplicity, to take at this point K and F both to be $\mathbb{Q}$. Once the ideas are clear, it is easy enough to transfer them to general F and K, but not necessary to do so in a review.

The expectation is that the order $n = n(k, M)$ of the zero of $L(z, M)$ at $z = k$, k an integer, will be expressible directly in terms of geometric and arithmetic properties of M, and so will

$$R(M) = \lim_{z \to k} \frac{L(z, M)}{(z - k)^n}. \tag{4}$$

These geometric and arithmetic properties are defined the mixed motives attached to the pure motive M. Mixed motives appear in the theory of L-functions as extensions of powers

$\mathbb{T}(m)$ of the Tate motive by M and in the simplest cases are determined by, say, divisors over the ground field F (for example, $\mathbb{Q}$) on a curve or, indeed, on any smooth projective variety over F. The most familiar examples are rational points on elliptic curves. Of importance are the extensions N of the form

$$0 \to M \to N \to \mathbb{T}(-k) \to 0. \tag{5}$$

as well as similar extensions in related categories defined by various cohomology theories for varieties, motives and mixed motives, by de Rham theories, by the Betti theory for varieties over the real and complex fields, and and p-adic theories that attach to the motive M a p-adic Galois representation of dimension equal to the rank of M.

The motive M has a weight $w(M)$, which if M is a piece of the cohomology of a smooth projective variety is the degree in which it appears. So, by the last of the Weil conjectures, for almost all finite places $\mathfrak{p}$ of F there are associated to M algebraic numbers $\alpha_1(\mathfrak{p}), \ldots, \alpha_d(\mathfrak{p})$ of absolute value $\mathrm{N}\,\mathfrak{p}^{w(M)/2}$. The integer d is the rank of M. Thus $L(z, M)$ which is essentially

$$\prod_{\mathfrak{p}} \frac{1}{\prod_1^d (1 - \alpha_i(\mathfrak{p})/\mathrm{N}\,\mathfrak{p}^z)}$$

does not vanish for $\Re z > w(M) + 1$.

Suppose that it can be analytically continued with a functional equation of the expected type, thus

$$\Gamma(z, M)L(z, M) = \epsilon(z, M)\Gamma(1 - z, \hat{M})L(1 - z, \hat{M}),$$

where $\epsilon(z, M)$ is a constant times an exponential in z and thus nowhere vanishing, $\hat{M}$ a dual motive, which will be of weight $-w(M)$ and of the same rank as M, and $\Gamma(z, M)$ a product of Γ-factors. The product is a product over the infinite places v of the basic field F. If the weights in the Hodge structure of the Betti cohomology associated to M at v are $\{(p_1, q_1), \ldots, (p_d, q_d)\}$ and v is complex, the Γ-factor is $\prod_i \Gamma(s - \min(p_i, q_i))$, if v is real it is $\prod_i \Gamma(s/2 + \epsilon_i/2 - \min(p_i, q_i)/2)$, where ϵ_i is either 0 or 1.

The center of the critical strip is $w(M)/2+1/2$. Suppose $w(M)$ is even, then for integral $k > w(M)/2 + 1$, $L(k, M) \neq 0$ and for integral $k > -w(M)/2 + 1$, $L(k, \hat{M}) \neq 0$. The functional equation allows us to deduce from this the order of the zero of $L(z, M)$ at all integral $k < w(M)/2$. Moreover the order of the pole of $L(k, M)$ at $w(M)/2 + 1$ is presumably equal to the multiplicity with which M contains the Tate motive $\mathbb{T}(-w(M)/2)$. Applying this to $\hat{M}$ we deduce the order of the zero of $L(z, M)$ at $w(M)/2$. So there is, in principle, no mystery about the order of the zero of $L(z, M)$ at any integer when $w(M)$ is even. When $w(M)$ is odd, the same arguments deal with all integral points except $w(M)/2 + 1/2$, but this point is very important, being for example the one appearing in the conjecture of Birch and Swinnerton-Dyer. So the order of vanishing of $L(z, M)$ at the BSD-point $z = w(M)/2 + 1/2$ is related to much more recondite geometric information. According to the conjectures of Beilinson and Deligne, the irrational factor of (4) is determined topologically by the motive over the infinite places of the field F (cf. [Ha, Fo]). We first consider $k \geq w(M)/2 + 1$, supposing that M does not contain the Tate motive $\mathbb{T}(-w(M)/2)$ as a factor.

The motive has, on the one hand, a Betti cohomology $H_{\mathrm{B}}(M)$ over $\mathbb{Q}$ that when tensored with $\mathbb{C}$ has a Hodge structure

$$H_{\mathrm{B}}(M)\otimes\mathbb{C}=\oplus_{p+q=w(M)}H^{p,q}(M)$$

and, on the other, a de Rham cohomology $H_{\mathrm{dR}}(M)$ over $\mathbb{Q}$ with a filtration

$$\dots F^{p-1}(M)\supset F^{p}(M)\supset F^{p+1}(M)\dots$$

that terminates above at $H_{\mathrm{dR}}(M)$ and below at 0. Moreover $H_{\mathrm{B}}(M)\otimes\mathbb{C}$ and $H_{\mathrm{dR}}(M)\otimes\mathbb{C}$, identified with de Rham cohomology over $\mathbb{C}$, are canonically isomorphic and under the isomorphism

$$F^{p}(M)\simeq\oplus_{w(M)\geq p}H^{p,q}(M).$$

There is an involution ι_1 on $H_{\mathrm{B}}(M)$ that that arises from the complex conjugation of varieties over $\mathbb{Q}$. It can be extended to $H_{\mathrm{B}}(M)\otimes\mathbb{C}$ linearly. There is a second involution $\iota_2: x\otimes z\to x\otimes\bar{z}$ on this tensor product. On the other hand, complex conjugation defines an involution ι of the de Rham cohomology over $\mathbb{C}$. Under the canonical isomorphism $\iota_1\circ\iota_2$ becomes ι. The particular pair $H_{\mathrm{B}}(M)$ and $H_{\mathrm{dR}}(M)$ with the auxiliary data described define a structure that we denote M_{HdR} but we can also consider the category of all such structures, referred to in [Ha] as the category of Hodge-de Rham structures and here as HdR-structures. This category also contains extensions

$$0\to M_{\mathrm{HdR}}\to N_{\mathrm{HdR}}\to\mathbb{T}_{\mathrm{HdR}}(-k), \tag{6}$$

in which N_{HdR} may not be associated to a motive. Nevertheless extensions (5) in the category of mixed motives presumably give rise to extensions (6) in the category of HdR-structures.

It follows readily from the definition of HdR-structures, mixed or not, that the sequence (6) splits if $k\leq w(M)/2$. Otherwise the group $\mathrm{Ext}^1_{\mathrm{HdR}}(\mathbb{T}_{\mathrm{HdR}}(-k),M_{\mathrm{HdR}})$ formed by classes of the extensions (6) can be calculated readily as

$$H_{\mathrm{dR}}(M)\otimes\mathbb{R}/\{H_{\mathrm{B}}(M)^{+}+F^{0}(M)\}.$$

The vector space $H_{\mathrm{B}}(M)^{+}$ is the plus eigenspace if ι_1 in $H_{\mathrm{B}}(M)$.

The group in the (hypothetical) category of mixed motives formed by classes of the extensions in (5) is denoted $\mathrm{Ext}^1(\mathbb{T}(-k),M)$. The functor $M\to M_{\mathrm{HdR}}$ leads to

$$\mathrm{Ext}^1(\mathbb{T}(-k),M)\to\mathrm{Ext}^1_{\mathrm{HdR}}(\mathbb{T}_{\mathrm{HdR}}(-k),M_{\mathrm{HdR}})\equiv H_{\mathrm{dR}}(M)\otimes\mathbb{R}/\{H_{\mathrm{B}}(M)^{+}+F^{0}(M)\}.$$

in which $H_{\mathrm{B}}(M)^{+}$ is the fixed point set of ι_1 The combined conjectures of Beilinson and Deligne affirm not only that the resulting map

$$\mathrm{Ext}^1(\mathbb{T}(-k),M)\to H_{\mathrm{dR}}(M)\otimes\mathbb{R}/\{H_{\mathrm{B}}(M)^{+}+F^{0}(M)\}$$

is injective but also that it yields an isomorphism of $\mathrm{Ext}^1(\mathbb{T}(-k), M)\otimes\mathbb{R}$ with the quotient on the right. Thus the product of the determinant of a basis of $\mathrm{Ext}^1(\mathbb{T}(-k), M)$ with the determinant of a basis of $H_{\mathrm{B}}(M)^+$ can be compared with the determinant of a basis of the rational vector space $H_{\mathrm{dR}}(M)/F^0(M)$, the quotient being an element of $\mathbb{C}^\times/\mathbb{Q}^\times$ that is supposed, as part of the Beilinson-Deligne complex of conjectures, to be the image of (4).

Although the notion of a mixed motive is somewhat uncertain and little has been proved, the theory is in fact strongly geometric with, I find, considerable intuitive appeal. Moreover when developed systematically, it permits a clean description of the integers n appearing in (4), even when k is the BSD-point, and of the limits $R(M)$, not simply up to a rational number as in the conjectures of Beilinson-Deligne, but precisely as in the conjectures of Bloch-Kato. Although clean, the description is neither brief nor elementary. It is expounded systematically in [FP].

The general form of the Main Conjecture of Iwasawa can also be profitably formulated in the context of mixed objects. Recall that part of Hida's program is to attach to π a p-adic representation in ${}^LG$ and thus to each representation r a p-adic family of representations σ_r. The principal objective of the book is the algebro-geometrical constructions that enable him to transfer to Siegel varieties, thus to the Shimura varieties associated to symplectic groups in higher dimensions, the techniques developed by him earlier for $GL(2)$ over $\mathbb{Q}$ and over totally real fields and to construct for them a theory of p-adic automorphic forms, from which a construction of p-adic L-functions might be deduced. This is a well-established tradition. The p-adic L-functions are constructed either directly as interpolating functions or indirectly from the Fourier expansions of p-adic automorphic forms and then the main conjecture affirms that they are equal to the characteristic function of a Selmer group defined by a parametrized family of p-adic Galois representation, essentially, if I am not mistaken, by showing that this characteristic function does interpolate the modified values of the complex automorphic L-function. The first, easiest, yet extremely difficult cases of the Riemann zeta-function and Dirichlet L-functions are in [MW].

The main conjecture could therefore be formulated directly in terms of the complex L-function and the p-adic representation were it not that, at present, the only way to construct the parametrized Galois representations is often, as in Hida's books and papers, through the mediating family of p-adic automorphic forms.

The p-adic space A on which the p-adic L-function was to be defined could – since we agreed to take both fields K and F to be $\mathbb{Q}$ – be the continuous $\mathbb{Q}_p$-valued spectrum of a commutative ring R over $\mathbb{Z}_p$, thus the continuous homomorphisms of R into $\mathbb{Q}_p$. The ring R will be chosen such that these homomorphisms all have image in $\mathbb{Z}_p$. R could be, for example, a power series ring over a module of finite rank over $\mathbb{Z}_p$. For example, the extension of $\mathbb{Q}$ generated by all p^nth roots of unity contains a subfield $\mathbb{Q}_\infty$ over which it is of finite index and for which $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is isomorphic to $\mathbb{Z}_p \equiv 1 + p\mathbb{Z}_p$. Let $\Lambda = \lim_{p\to\infty} \mathbb{Z}_p(\Gamma')$ be the limit over finite quotients of Γ. Of course $1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. The ring Λ is a common choice of R and is isomorphic to a power series ring $\mathbb{Z}_p[[T]]$. Its continuous $\mathbb{Q}_p$-valued spectrum may be identified with the continuous homomorphisms of Γ into $\mathbb{Q}_p^\times$.

Certain isolated points λ in the spectrum of R were to correspond to motives $M(\lambda)$. If the primary object is not the p-adic L-function but a family $\{\sigma_s\}$ of p-adic representations over R, say into $GL(d,\mathbb{Q}_p)$ then at $s=\lambda$, the representation $\sigma_{M(\lambda)}=\lambda\circ\sigma$ is to be that associated to $M(\lambda)$, thus that on the p-adic étale cohomology $H_p(M(\lambda))$.

The p-adic representation σ_M of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the étale cohomology $H_p(M)$ of a motive M over $\mathbb{Q}$ or perhaps better the restriction of σ_M to $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is an object whose theory ([FI]) I do not yet understand and do not try to describe. Perhaps the most important thing to recall is that its Hodge type, which describes the action on the tensor product of $H_p(M)$ with the completion $\mathbb{C}_p$ of $\bar{\mathbb{Q}}_p$, is a sequence of integers $h_1,\dots,h_d$, with d equal to the dimension of M, supposed pure.

In [Gr] very tentative, yet very appealing conjectures are formulated. They are difficult to understand, but are a benchmark with which to compare the aims and results of Hida. First of all, the representation σ is supposed to take values in $GL(d,R)$. Then the parametrized repesentations arise on taking a continuous homomorphism $\phi=\phi_s : R\to\mathbb{Z}_p$, $s\in A$ and composing it with σ.

Denote the space of the representation σ by $V=R^d$. The appropriate analogue for p-adic representations of the mixed objects (6) would appear at first to be extensions

$$0\to V\to W\to T\to 0, \tag{7}$$

in which $T=T(0)$ is the one-dimensional trivial representation, so that W stands for a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of degree $d+1$. Thus k in (7) has been taken to be 0, a formal matter because the sequence can be twisted. If we write the representation on W in block form, the first diagonal block $d\times d$ and the second 1×1, only the upper-diagonal $d\times 1$ block is not determined and it defines an element of $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),V)$.

Not this group appears in [Gr] but the group

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\tilde{V}),\qquad \tilde{V}=V\otimes\mathrm{Hom}(R,\mathbb{Q}_p/\mathbb{Z}_p)). \tag{8}$$

More precisely, it is a subgroup of this group, the Selmer group S, that is pertinent. It is defined as an intersection over primes q of subgroups defined by local conditions. If $q\neq p$ the subgroup is the kernel of the restriction to the decomposition group. To define the subgroup at $q=p$, Greenberg imposes a condition that he calls the Panchiskin condition, a condition that I do not understand, although the notion of an ordinary form or Galois representation seems to be an expression of it.

Thus the group S is defined by extensions that are a reflection at the p-adic level of extensions of motives. R acts on it and on its dual $\hat{S}=\mathrm{Hom}(S,\mathbb{Q}_p/\mathbb{Z}_p)$. The general form of the main conjecture would be that the characteristic ideal of $\hat{S}$, an element in the free abelian group on the prime ideals of R of height one, is – apart from some complications related to those that arose at $k=w(M)/2+\epsilon$, $\epsilon=0,1,2$ – essentially the interpolating p-adic L-function. This is vaguely expressed both by Hida and Greenberg and even more vaguely by me, because I understand so little, but, as a general form of the Main Conjecture of Iwasawa, it is, in concert with the Fontaine/Perrin-Riou form of the Beilinson-Deligne-Bloch-Kato conjectures, of tremendous appeal.

As a valediction I confess that I have learned a great deal about automorphic forms while preparing this review, but not enough. It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates. To see it whole is certainly a daunting, for the moment even impossible, task. Obtaining proofs of serious results is another, even more difficult matter and each success demands an enormous concentration of forces.

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