New Haven

September 22, 1969

Dear Harish-Chandra,

I am very pleased that you are willing to come here to speak. I have decided to come to Princeton on Monday, October 7 and to stay over till Tuesday. However I don't want to give a lecture then because Weil and Jacquet are familiar with anything conclusive I did before January and I have not yet finished the things I have been doing since. I have already given you an idea what these things are but I can perhaps be more precise now.

If F is a local field let C_F be the multiplicative group of F and if F is a global field let C_F be the idèle class group of F. As I said before if $K\backslash F$ is normal the Weil group $G_{K|F}$ is an extension of C_F by the Galois group of $K\backslash F$. If one likes one can take projective limits and get an object called the Weil group of F. If, F is global, and $F_\mathfrak{p}$ is a completion of F, and $K_\mathfrak{P}$ is a completion of K over $\mathfrak{p}$ then there is a map of $G_{K_\mathfrak{P}}\backslash F_\mathfrak{P}$ into $G_{K_\mathfrak{P}}\backslash F_\mathfrak{p}$. A representation ρ is a finite-dimensional representation over $\mathbb{C}$ such that $\rho(g)$ is semi-simple for all g.

For a non-archimedean local field I can attach to every equivalence class ω of representations a function $L(s,\omega)$ in just one way so that the following conditions are satisfied.

(i) If $\omega \sim \chi_F$ is one dimensional

$$\begin{aligned} L(s,\omega) &= \tfrac{1}{1-\chi_F(\pi_F)|\pi_F|^s} && \text{if } \chi_F \text{ is trivial on units} \\ &= 1 && \text{if } \chi_F \text{ is not trivial on units.} \end{aligned}$$

(ii) $L(s,\omega_1 \oplus \omega_1) = L(s,\omega_1)L(s,\omega_2)$.

(iii) If $\omega_1 \simeq$ Ind $(G_{K\backslash F}, G_{K\backslash E}, \omega_2)$ then

$$L(s,\omega_1) = L(s,\omega_2).$$

For archimedean fields $L(s,\omega)$ is defined by the following conditions

(i) If $F = \mathbb{R}$ and $\omega \sim \chi_\mathbb{R}$ with $\chi_\mathbb{R}(x) = (\text{sgn } x)^m |x|^r, m = 0$ or 1, then

$$L(s,\omega) = \pi^{-\frac{1}{2}(s+r+m)}\Gamma\left(\frac{s+r+m}{2}\right).$$

(ii) If $F = \mathbb{C}$ and $\omega \sim \chi_\mathbb{C}$ with $\chi_\mathbb{C}(z) = |z|^{2r}\frac{z^n \bar{z}^n}{|z|^{m+n}}, m+n \geq 0, mn = 0$ then

$$L(s,\omega) = 2(2\pi)^{-(s+r+\frac{m+n}{2})}\Gamma(s+r+\frac{m+n}{2})$$

(iii) $L(s,\omega_1 \oplus \omega_2) = L(s,\omega_1)L(s,\omega_2)$

(iv) If $\omega_1 \simeq$ Ind $(G_{K\backslash F}, G_{K\backslash E}, \omega_2)$ then

$$L(s,\omega_1) = L(s,\omega_2).$$

If K is a global field, ω is an equivalence class of representations of the Weil group and $\omega_\mathfrak{p}$ the induced equivalence class of the Weil group of $F_\mathfrak{p}$ the Artin L-function is

$$L(s,\omega)=\Pi_\mathfrak{p} L(s,\omega_\mathfrak{p})$$

Let me state the theorem which I have been working on since last January and then comment on its relation to the functional equations of the L-functions. By the way, as soon as I understand a lemma of Dwork I should be able to write out the proof of this theorem. I think I have all the parts of the proof except for some easily managed details. If $F=\mathbb{R}$ and $\chi_\mathbb{R}(x)=(\mathrm{sgn}\ x)^m|x|^r$ with $m=0$ or 1 and $\psi_\mathbb{R}$ is the additive character $\psi_\mathbb{R}(x)=e^{2\pi iux}$ I set

$$\Delta(\chi_\mathbb{R},\psi_\mathbb{R})=(i\ \mathrm{sgn}\ u)^m|u|^r.$$

If $F=\mathbb{C}$ and $\chi_\mathbb{C}(z)=|z|^{2r}\frac{z^m\bar{z}^n}{|z|^{m+n}}, m+n\geq 0, mn=0$ and $\psi_\mathbb{C}$ is the additive character $\psi_\mathbb{C}(z)=e^{4\pi i\Re(wz)}$. I set

$$\Delta(\chi_\mathbb{C},\psi_\mathbb{C})=i^{m+n}\chi_\mathbb{C}(w).$$

If F is non-archimedean, if χ_F is a generalized character of C_F with conductor $\mathfrak{P}_F^m$, if ψ_F is a non-trivial additive character of F and $\mathfrak{P}_F^{-n}$ is the largest ideal on which it vanishes and $O_F\gamma=\mathfrak{P}_F^{m+n}$ I set

$$\Delta(\chi_F,\psi_F)=\chi_F(\gamma)\frac{\int_{U_F}\psi_F(\frac{\alpha}{\gamma})\chi_F^{-1}(\alpha)d\alpha}{\left|\int_{U_F}\psi_F(\frac{\alpha}{\gamma})\chi_F^{-1}(\alpha)d\alpha\right|}.$$

U_F is the group of units. The right side is independent of γ.

If E is a separable extension of F and ψ_F is given then

$$\psi_{E\backslash F}(X)=\psi_F(T_{r_{E\backslash F}}X).$$

Theorem. *Suppose F is a given local field and ψ_F a given non-trivial additive character of F. It is possible in exactly one way to assign to each separable extensions E of F a complex number $\rho(E\mid F,\psi_\gamma)$ and to each equivalence classic of representations of the Weil group of E a complex number $\epsilon(\omega,\psi_{E|F})$ so that*

(i) *If $\omega\simeq\chi_E$ then $\epsilon(\omega,\psi_{E|F})=\Delta(\chi_E,\psi_{E|F})$*

(ii) $\epsilon(\omega_1\oplus\omega_1,\psi_{E|F})=\epsilon(\omega_1,\psi_{E|F})\epsilon(\omega_2,\psi_{E|F})$

(iii) *If $\omega_1\simeq$ Ind $(G_{K\backslash F},G_{K\backslash E},\omega_2)$ then*

$$\epsilon(\omega_1,\psi_F)=\rho(E\mid F,\psi_F)^{\dim\omega_2}\epsilon(\omega_2,\psi_{E|F}).$$

If A_F^s is the generalized character $\alpha \to |\alpha|_F^s$ set $\epsilon(s,\omega,\psi_F) = \epsilon(A_F^{s-1/2} \otimes \omega, \psi_F)$. If F is a global field and ψ_F a non-trivial character of $\mathbb{A}_F/F$ let $\psi_{F_\mathfrak{p}}$ be the restriction of ψ_F to $F_\mathfrak{p}$. If ω is an equivalence class of representations of the Weil group of F and

$$\epsilon(s,\omega) = \Pi_\mathfrak{p}\epsilon(s,\omega_\mathfrak{p},\psi_{F_\mathfrak{p}})$$

the functional equation of the L-function can, on the basis of the previous theorem, be shown to be

$$L(s,\omega) = \epsilon(s,\omega)L(1-s,\tilde{\omega})$$

if $\tilde{\omega}$ is contragredient to ω.

Once I have this theorem I should very quickly be able to derive a consequence of interest for group representations. However I have not yet carried out the computations. After some preliminaries I shall state the consequence.

If F is a non-archimedean local field a two-dimensional equivalence class ω of representations of the Weil group of F will be called special if ω is the direct sum of two one-dimensional representations μ_F and ν_F and $\mu_F\nu_F^{-1} = A_F^1$ or A_F^{-1}. If $F = \mathbb{R}$ a two-dimensional equivalence class ω is special if $\omega \simeq \mu_F \oplus \nu_F$, and $\mu_F(x) = |x|^{s_1}\left(\frac{x}{|x|}\right)^{m_1}, \nu_F(x) = |x|^{s_1}\left(\frac{x}{|x|}\right)^{m_2}$ and $(s_1-s_2)-(m_1-m_2)$ is an odd integer. If $F = \mathbb{C}$, ω is special if $\omega \simeq \mu_F \oplus \nu_F$

$$\mu_F(z) = |z|^{2s_1}\left(\frac{z}{|z|}\right)^{m_1}$$
$$\nu_F(z) = |z|^{2s_2}\left(\frac{z}{|z|}\right)^{m_2}$$

and one of $\frac{s_1-s_2}{2} - \left(1+\frac{|m_1-m_2|}{2}\right)$ and $\frac{s_2-s_1}{2} - \left(1+\frac{|m_1-m_2|}{2}\right)$ is a non-negative integer. $L(\psi_F)$ is the space of functions on $GL(2,F)$ satisfying

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \equiv \psi_F(x)\varphi(g)$$

together with certain conditions on smoothness and rate of growth. Here is the consequence I mentioned.

Theorem. *Suppose that for every global field F and every two-dimensional irreducible equivalence class of representations of the local group of F the function $L(s,\omega)$ is entire and bounded in vertical strips. Then if F is a local field, ω a two-dimensional equivalence class of representations of the Weil group of F which is not special, and ψ_F a non-trivial additive character of F there is a unique simple representation π_ω of $GL(2,F)$ satisfying*

(i) π_ω acts on $L \subseteq L(\psi_F)$

(ii) If φ belongs to L and χ_F is a generalized character of C_F the integral

$$\int_{C_F} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)\chi_F(\alpha)|\alpha|_F^s d\alpha$$

converges for Re s sufficiently large. Denote its value by $\Phi(g,s,\varphi,\chi_F)$ and set

$$\Phi(g,s,\varphi,\chi_F) = L(s+\frac{1}{2},\omega\otimes\chi_F)\Phi'(g,s,\varphi,\chi_F)$$

$\Phi'(g,s,\varphi,\chi_F)$ is an entire function of s bounded in vertical strips. Moreover

$$\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g; -s, \varphi(\det\omega\chi_F)^{-1}\right) = \epsilon(\chi_F A_F^s \otimes \omega, \psi_F)\Phi'(g,s,\varphi,\chi_F)$$

if $\det\,\omega$ is the 1-dimensional representation obtained from ω by taking determinants.

It will follow that π_{ω_1} equivalent to π_{ω_2} implies $\omega_1 = \omega_2$. This theorem makes the existence of the representations I mentioned to you earlier pretty much certain. If they do not exist the L-series behave in an entirely unexpected manner.

Yours truly,

Bob Langlands